The entropy of directed random walks on infinite rooted self-similar graphs

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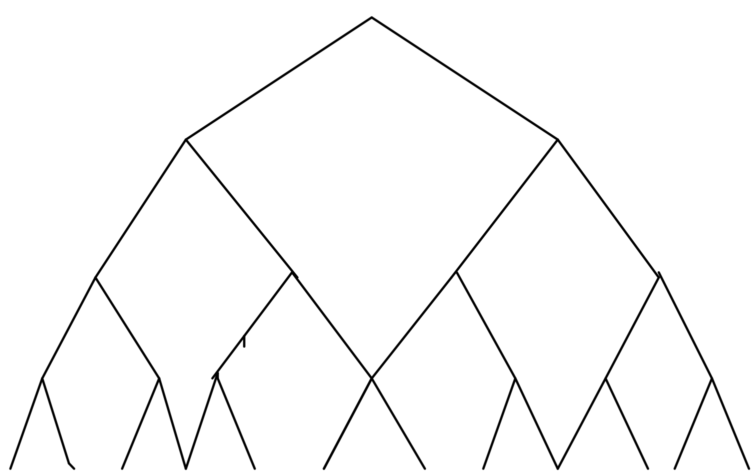

Figure 1: The Fibonacci graph

Construction of the graphs

Consider the set of finite words $\mathbf{A}^{(\mathbb{N})}$ and an equivalence relation $\simeq$ with

$$v \simeq u \Rightarrow l(v) = l(u)$$

$$v \simeq u \wedge p \simeq w \Rightarrow vp \simeq uw$$

$$v \simeq u \wedge vp \simeq uw \Rightarrow p \simeq w$$

The vertices of the graph are:

$$\mathbf{V} := \mathbf{A}^{(\mathbb{N})}/_{\simeq} = \{[w] | w \in \mathbf{A}^{(\mathbb{N})}\}.$$

Two vertices $[w]$ and $[v]$ are connected by an edge if and only if for a symbol $i \in \mathbf{A}$

$$vi \in [w] \text{ or } wi \in [v].$$

By this construction we achieve

$$\mathbf{V} \cong \mathbf{V}[v] = \{[uw] | u \in [v] \wedge w \in \mathbf{A}^{(\mathbb{N})}\}$$

and the exponential growth rate $H(\mathbf{V})$ exists

$$H(\mathbf{V}) := \lim_{n \longmapsto \infty} \frac{\log Card(\mathbf{V}_n)}{n}$$

since $Card(\mathbf{V}_n)$ is subadditive.

Directed Random walks and their entropy

On the space of infinite words $\mathbf{A}^{\mathbb{N}}$ consider an invariant Borel probability measure μ.
For a word $w \in \mathbf{A}^{(\mathbb{N})}$ we have the cylinder set

$\langle w \rangle := \{(v_i) \in \mathbf{A}^{\mathbb{N}} | (v_1, \ldots, v_n) = (w_1, \ldots w_n)\}.$

For a vertex $[w] \in \mathbf{V}$ we define the cylinder

$$\langle [w] \rangle = \bigcup_{v \simeq w} \langle v \rangle.$$

The probability to reach a vertex [w] from the root is

$$\mu([w]) = \mu(\langle [w] \rangle).$$

Consider the partition induced by the vertices in $\mathbf{V}_n$

$$\Pi_n = \{\langle [w] \rangle | [w] \in \mathbf{V}_n\}.$$

Define the entropy of the random walk by

$$h(\mu, \mathbf{V}) := \lim_{n \longmapsto \infty} \frac{H(\mu, \Pi_n)}{n}$$

The limit exists due to self-similarity of tree and invariance of the measure.

Results on the entropy

Since the partition into usual cylinders is finer than the partition into cylinders of vertices

$$h(\mu, \mathbf{V}) \leq h(\mu)$$

Equality holds only for the full tree. Similar to usual metric entropy

$$\mu \mapsto h(\mu, \mathbf{V})$$

is upper-semi continues and affine.
Since $Card\Pi_n = Card\mathbf{V}_n$ we have

$$h(\mu, \mathbf{V}) \leq H(\mathbf{V}).$$

Theorem *Every rooted infinite self-similar graph has a random walk* μ *with full entropy*

$$h(\mu, \mathbf{V}) = H(\mathbf{V})$$

The measure μ *may be chosen ergodic.*

Theorem *If* b *is a Bernoulli measure*

$$h(b, \mathbf{V}) = \lim_{n \longmapsto \infty} \frac{1}{n} - \log b([w_1, \ldots, w_n])$$

for almost all $w \in \mathbf{A}^{\mathbb{N}}$.

Limit measures

Project the measure μ on $\mathbf{V_n}$ equidistant to an interval

$$\mu_n = \mu \circ \pi_n^{-1}$$

Let a limit measure $\mu_{\mathbf{V}}$ of a random walks on a self-similar tree $\mathbf{V}$ be a weak* accumulation point of the sequence μ_n.

Theorem *Every infinite rooted self-similar graph with exact growth rate has a random walk with absolutely continuous limit measures.*

Idea: If one limit measure $\mu_{\mathbf{V}}$ is not absolutely continuous with respect to the Lebesgue measure one proves $h(\mu, \mathbf{V}) < H(\mathbf{V})$.

Questions: Are the limit measures $\mu_{\mathbf{V}}$ unique? What's about their dimension or their density?