

# Results on self-similar measures on the real line

Acta Mathematica Hungarica, vol. 92 (1-2), 143-161, 2001  
Acta Mathematica Hungarica, vol. 113 (4-4), 333-343, 2006  
and one recent paper accepted by Fractals

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A generalization of the work of J. Peres, K.  
Simon and B. Solomyak

## Construction of the measures

For contraction rates  $B = (\beta_1, \dots, \beta_n) \in (0, 1)^n$  and  $D = (d_1, \dots, d_n) \in \mathbb{R}^n$  translations consider an IFS

$$T_i x = \beta_i x + d_i \quad i = 1, \dots, n$$

For a probability vector  $P = (p_1, \dots, p_n)$  there is a unique self-similar measure  $\mu_{B,D}^P$  with

$$\mu_{B,D}^P = \sum_{i=1}^n p_i T_i(\mu_{B,D}^P)$$

$$\mu_{B,D}^P = \pi_{B,D}(b^P)$$

where  $b^P$  is a Bernoulli measure on the sequence space  $\Sigma$  and

$$\pi_{B,D}(s) = \sum_{k=0}^{\infty} s_k \prod_{i=1}^n \beta_i^{\#_i^k(s)}$$

$$\#_i^k(s) = \text{Card}\{s_j | s_j = d_i \text{ for } j = 0, \dots, k-1\}.$$

## Singularity

$$\dim \mu_{B,D}^P \leq \frac{-\sum_{i=1}^n p_i \log(p_i)}{-\sum_{i=1}^n p_i \log(\beta_i)}$$

Hence  $\mu_{B,D}^P$  is singular if

$$\prod_{i=1}^n p_i^{p_i} > \prod_{i=1}^n \beta_i^{p_i}.$$

**Idea:** The coding map is Lipschitz with respect to the metric

$$d(s, t) = \prod_{i=1}^n \beta_i^{\#\mathfrak{k}(s)}$$

with  $\mathfrak{k} = \min\{k | s_k \neq t_k\}$  and does not increase dimension. Use Birkhoff's and Shannon's theorem in the symbolic space  $(\Sigma, d)$  to get the right hand side.

# Transversality

Consider powerseries

$$\mathcal{F}_b = \left\{ f(x) = 1 + \sum_{k=1}^{\infty} b_k x^k \mid b_k \in [-b, b] \right\}$$

and let

$$t(b) = \min\{x > 0 \mid \exists f \in \mathcal{F}_b f(x) = f'(x) = 0\}$$

Each function  $f \in \mathcal{F}_b$  crosses the  $x$ -axis transversely with slope in  $[-\rho, \rho]$  on the interval of transversality  $[0, t(b) - \epsilon]$

It is known that

$$t(1) = 0.64913\dots$$

$$t(2) = 0.5$$

$$t(3) = 0.42772\dots$$

$$t(b) \geq (\sqrt{b} + 1)^{-1} \text{ for } b \in [1, 3 + \sqrt{8}]$$

$$t(b) = (\sqrt{b} + 1)^{-1} \text{ for } b \in [3 + \sqrt{8}, \infty).$$

## Generic absolute continuity

For all  $A \in (0, 1]^n$  and for almost all

$$\beta \in \left( \prod_{i=1}^n \left( \frac{p_i}{\alpha_i} \right)^{p_i}, t(D, A) \right)$$

the self-similar measure  $\mu_{\beta A, D}^P$  is absolutely continuous and has a density in  $L^2$  for almost all

$$\beta \in \left( \sum_{i=1}^n \frac{p_i^2}{\alpha_i}, t(D, A) \right).$$

The lower bound on a.c. is sharp.

The upper bound is given by the transversality techniques

$$b = \frac{\max\{\alpha_i d_j \mid d_j > 0\} + \max\{-\alpha_i d_j \mid d_j < 0\}}{\min_{i \neq j} |d_i - d_j|}$$

$$t(D, A) = t(b) - \epsilon.$$

## One example

Let  $D \in (0, 2]^n$  and  $P = (1/n, 1/n, \dots, 1/n)$  and  $A = (\alpha_1, \alpha_2, \dots, \alpha_n)$ .

We get absolute continuity of the corresponding self-similar measure  $\mu_{\beta A, D}^P$  for almost all

$$\beta \in (1/(n \sqrt[n]{\alpha_1 \alpha_2 \dots \alpha_n}), 0.5)$$

and density in  $L^2$  for almost all

$$\beta \in (1/n^2 \sum \frac{1}{\alpha_i}, 0.5)$$

$A = (1, 2/3, 3/4)$ . For almost all

$$\beta \in (1/(3 \sqrt[3]{2}), 0.5)$$

the self-similar measure  $\mu_{B, D}^P$  with

$$B = (\beta, 2\beta/3, 3\beta/4)$$

is absolutely continuous and has a density in  $L^2$  for almost

$$\beta \in (23/54, 1/2)$$

## Number theoretical exceptions

Consider the special case

$$T_1x = \beta_1x \quad T_2x = \beta_2x + 1$$

With  $P = (1/2, 1/2)$  for almost all  $\beta_1, \beta_2 \in (0, 0.649)$  with  $\beta_1\beta_2 \geq 1/4$  the self-similar measures  $\mu_{\beta_1, \beta_2}$  are absolutely continuous.

For all  $\beta_2 \in (1/4, 1/2)$  and  $\epsilon > 0$  sufficient small there is an

$$\beta_1 \in \left(\frac{1}{4\beta_2}, \frac{1}{4\beta_2} + \epsilon\right)$$

such that the measure  $\mu_{\beta_1, \beta_2}$  is singular with  $\dim_H \mu_{\beta_1, \beta_2} < 1$ .

One example is given by a solution  $\beta_1, \beta_2$  of

$$\begin{aligned} & -x^2 + x^2y + x^2y^2 + x^2y^4 - x^3y^4 \\ & = xy - x^2y - x^3y - x^4y - x^5y + x^5y^2 \end{aligned}$$