Results on self-similar measures on the real line

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and one recent paper accepted by Fractals

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A generalization of the work of J. Peres, K. Simon and B. Solomyak

Construction of the measures

For contraction rates $B = (\beta_1, \ldots, \beta_n) \in (0,1)^n$ and $D = (d_1, \ldots, d_n) \in \mathbb{R}^n$ translations consider an IFS

$$T_i x = \beta_i x + d_i \qquad i = 1, \ldots n$$

For a probability vector $P = (p_1, \ldots, p_n)$ there is a unique self-similar measure $\mu^P_{B,D}$ with

$$\mu^P_{B,D} = \sum_{i=1}^{n} p_i \, T_i(\, \mu^P_{B,D} \,)$$

$$\mu^P_{B,D} = \pi_{B,D}(b^P)$$

where b^P is a Bernoulli measure on the sequence space Σ and

$$\pi_{B,D}(s) = \sum_{k=0}^{\infty} s_k \prod_{i=1}^{n} \beta_i^{\sharp_i^k(s)}$$

$\sharp_i^k(s) = Card\{s_j | s_j = d_i \text{ for } j = 0, \ldots, k-1\}$.

Singularity

$$\dim \mu_{B,D}^{P} \leq \frac{-\sum_{i=1}^{n} p_i \log(p_i)}{-\sum_{i=1}^{n} p_i \log(\beta_i)}$$

Hence $\mu_{B,D}^{P}$ is singular if

$$\prod_{i=1}^{n} p_i^{p_i} > \prod_{i=1}^{n} \beta_i^{p_i}.$$

Idea: The coding map is Lipschitz with respect to the metric

$$d(s,t) = \prod_{i=1}^{n} \beta_i^{\sharp_i^{\mathfrak{k}}(s)}$$

with $\mathfrak{k} = \min\{k | s_k \neq t_k\}$ and does not increase dimension. Use Birkhoff's and Shannon's theorem in the symbolic space (Σ, d) to get the right hand side.

Transversality

Consider powerseries

$$\mathcal{F}_b = \{f(x) = 1 + \sum_{k=1}^{\infty} b_k x^k | b_k \in [-b, b]\}$$

and let

$$t(b) = \min\{x > 0 | \exists f \in \mathcal{F}_b \, f(x) = f'(x) = 0\}$$

Each function $f \in \mathcal{F}_b$ crosses the x-axis transversely with slope in $[-\rho, \rho]$ on the interval of transversality $[0, t(b) - \epsilon]$

It is known that

$$t(1) = 0.64913...$$

$$t(2) = 0.5$$

$$t(3) = 0.42772...$$

$$t(b) \geq (\sqrt{b} + 1)^{-1} \text{ for } b \in [1, 3 + \sqrt{8}]$$

$$t(b) = (\sqrt{b} + 1)^{-1} \text{ for } b \in [3 + \sqrt{8}, \infty).$$

Generic absolute continuity

For all $A \in (0,1]^n$ and for almost all

$$\beta \in \left(\prod_{i=1}^{n} \left(\frac{p_i}{\alpha_i}\right)^{p_i} ,\ t(D,A) \right)$$

the self-similar measure $\mu^P_{\beta A, D}$ is absolutely continuous and has a density in L^2 for almost all

$$\beta \in \left(\sum_{i=1}^{n} \frac{p_i^2}{\alpha_i} ,\ t(D,A) \right).$$

The lower bound on a.c. is sharp.

The upper bound is given by the transversality techniques

$$b = \frac{\max\{\alpha_i d_j | d_j > 0\} + \max\{-\alpha_i d_j | d_j < 0\}}{\min_{i \neq j} |d_i - d_j|}$$

$$t(D,A) = t(b) - \epsilon.$$

One example

Let $D \in (0,2]^n$ and $P = (1/n, 1/n, \ldots, 1/n)$ and $A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

We get absolute continuity of the corresponding self-similar measure $\mu^P_{\beta A, D}$ for almost all

$$\beta \in (1/(n\sqrt[n]{\alpha_1\alpha_2\ldots\alpha_n}), 0.5)$$

and density in L^2 for almost all

$$\beta \in (1/n^2 \sum \frac{1}{\alpha_i}, 0.5)$$

$A = (1, 2/3, 3/4)$. For almost all

$$\beta \in (1/(3\sqrt[3]{2}), 0.5)$$

the self-similar measure $\mu^P_{B,D}$ with

$$B = (\beta, 2\beta/3, 3\beta/4)$$

is absolutely continuous and has a density in L^2 for almost

$$\beta \in (23/54, 1/2)$$

Number theoretical exceptions

Consider the special case

$$T_1 x = \beta_1 x \qquad T_2 x = \beta_2 x + 1$$

With $P = (1/2, 1/2)$ for almost all $\beta_1, \beta_2 \in (0, 0.649)$ with $\beta_1 \beta_2 \geq 1/4$ the self-similar measures μ_{β_1, β_2} are absolutely continuous.

For all $\beta_2 \in (1/4, 1/2)$ and $\epsilon > 0$ sufficient small there is an

$$\beta_1 \in (\frac{1}{4\beta_2}, \frac{1}{4\beta_2} + \epsilon)$$

such that the measure μ_{β_1, β_2} is singular with $\dim_H \mu_{\beta_1, \beta_2} < 1$.

One example is given by a solution β_1, β_2 of

$$-x^2 + x^2 y + x^2 y^2 + x^2 y^4 - x^3 y^4$$
$$= xy - x^2 y - x^3 y - x^4 y - x^5 y + x^5 y^2$$