

From number theory to dynamical systems via geometric measure theory

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Outline

- ▶ Algebraic number theory: Pisot numbers

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- ▶ Geometric measure theory: Self-similar measures and Bernoulli convolutions
- ▶ Ergodic theory: Existence of ergodic measures of full dimension

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$$\left| \sum_{k=1}^n s_k \alpha^{-k} - \sum_{k=1}^n \bar{s}_k \alpha^{-k} \right| \geq C \alpha^{-n}$$

for a constant $C > 0$ and all $s_k, \bar{s}_k = \pm 1$ such that the points are different. [Garsia]

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- ▶ The set of Pisot numbers S is closed in \mathbb{R} . [Salem]

Self-similar measures

- ▶ Let (T_1, \dots, T_n) be linear contractions on an closed interval $I \subseteq \mathbb{R}$

$$T_i x = \beta_i x + d_i$$

and let (p_1, \dots, p_n) be a probability vector.

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- ▶ μ is either singular $\mu \perp \ell$ or equivalent $\mu \sim \ell$ to Lebesgue measure ℓ . [Bisbas/Ne.]

Bernoulli convolutions

- ▶ The infinite convolved Bernoulli measure

$$\mu = \star_{i=0}^{\infty} (p\delta_{\beta^i} + (1-p)\delta_{-\beta^i})$$

is the self similar measure with respect to the maps

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$$\beta \in [p^p(1-p)^{1-p}, 1)$$

and $\mu \perp \ell$ if β is not in this interval

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- ▶ This result can be generalized to inhomogeneous convolutions

$$\mu = \star_{i=0}^n (p_i\delta_{\beta^i} + (1-p_i)\delta_{-\beta^i})$$

[Bisbas/Ne.]

The Erdős-Salem theorem

- ▶ The Fourier transform of a (inhomogeneous) Bernoulli convolution μ is

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- ▶ For $p = 1/2$ "if" by [Salem] "only if" by [Erdős] and the dimension drop by [Alexander and York].
For $p \neq 1/2$ constant "only if" by [Lalley].
The version stated here [Bisbas/Ne.]

Nonuniform self-similar measures

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- ▶ There are solutions $\beta_1, \beta_2 \in (0, 0.66)$ of the algebraic equation
$$x - xy - xy^2 - xy^3 - xy^4 - xy^5 = -y + xy + x^2y + x^3y + x^4y + x^5y$$
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- ▶ Using Mathematica for $(\beta_1, \beta_2) = (0.39, 0.6432\dots)$ the measure μ is singular with $\dim_H \mu \leq 0.9986$.

[NE.]

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- ▶ If T is conformal with an repeller Λ than there exists an ergodic measure of full dimension on Λ . Especially this hold for holomorphic maps on \mathbb{C} and differentiable maps on \mathbb{R} .
[Bowen, Ruelle, Falconer]

Baker transformation

- ▶ The Baker transformation $f_\beta : \mathbb{R} \times [-1, 1] \mapsto \mathbb{R} \times [-1, 1]$ for $\beta \in (0, 1)$ is given by

$$f_\beta(x, y) = \begin{cases} (\beta x + (1 - \beta), 2y - 1) & \text{if } y \geq 0 \\ (\beta x - (1 - \beta), 2y + 1) & \text{if } y < 0 \end{cases}$$

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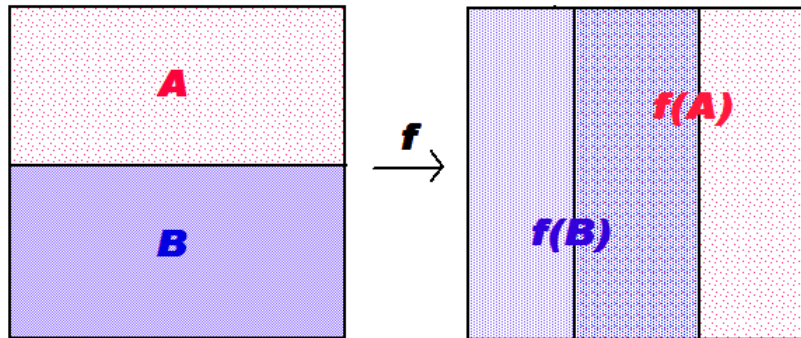
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- ▶ Let μ_β be the Bernoulli convolution with $p = 1/2$. The measure $\mu_\beta \times \ell$ is ergodic on Λ for f_β .

Figure I: The Baker transformation



Dimension theory for the Baker transformation

- ▶ If $\beta \in (0, 1/2)$ we have $\Lambda = C \times [-1, 1]$ for a Cantor set C and

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- ▶ If $\beta \in [1/2, 1)$ is the reciprocal of a Pisot number

$$\sup\{\dim_H \mu \mid \mu \text{ ergodic}\} < 2$$

using Erdős/Salem theorem and Garsia lemma. [Ne.]

Linear Solenoids

- For $\beta_1, \beta_2, \tau_1, \tau_2 \in (0, 1)$ let $f : [-1, 1]^3 \mapsto [-1, 1]^3$ be given by

$$f(x, y, z) = \begin{cases} (\beta_1 x + (1 - \beta_1), \tau_1 y + (1 - \tau_1), 2z - 1) & \text{if } y \geq 0 \\ (\beta_2 x - (1 - \beta_2), \tau_2 y - (1 - \tau_2), 2z + 1) & \text{if } y < 0 \end{cases}$$

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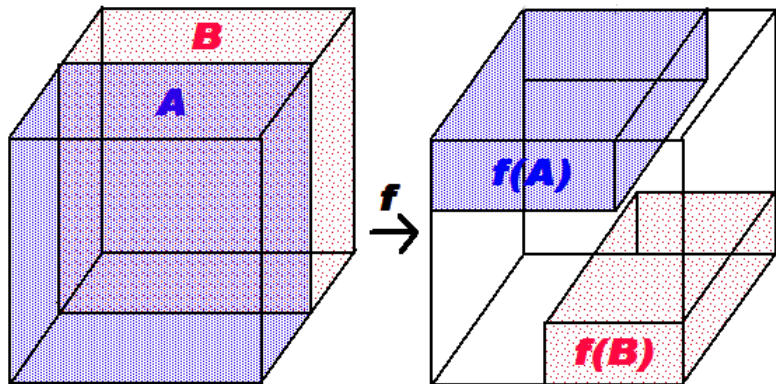
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- ▶ The projection of ergodic Bernoulli measures μ on Λ to the x -axis are non-uniform self similar measures.

Figure II: The linear solenoid transformation



Dimension theory for linear Solenoids

- ▶ For almost all $\beta_1, \beta_2 \in (0, 0.66)$

$$\dim_H \Lambda = D$$

where D is the solution

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