Multiscaling Behaviour of Cascade measures -a short survey containing open problems-

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Abstract

We give a short overview about some known results on the dimension and multifractal analysis of random and deterministic cascade measures. Furthermore we discuss the role of cascade measures in cascade and multifractal models of turbulence. By the way we point out some problems in the field that should be solved.

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1 Dimension and multifractal analysis of measures

Let $X \subset I\!R^j$ be compact and let $M(X)$ be the space of all Borel probability measures on X.

If $\mu \in M(X)$ is not absolutely continuous with respect to Lebesgue measure it may have a complicated "fractal" geometrical structure. The most important quantity that characteristics the concentration of the measure is its Hausdorff dimension[2].

Definition 1.1 Given $\mu \in M(X)$ the quantity

$$\dim_H \mu = \inf\{\dim_H B \mid B \text{ a Borel subset of } K \text{ and } \mu(B) = 1\}$$

is called the **Hausdorff dimension** of μ

The key fact used when determining is the following proposition (see [7], [28], [32] and [5]).

Proposition 1.1 *For $\mu \in M(X)$ we have*

$$\dim_H \mu = ess \sup_{\mu} \lim_{\epsilon \longmapsto 0} \inf \frac{\mu(B_\epsilon(x))}{\log \epsilon} = \sup\{\rho | \mu\{x| \lim_{\epsilon \longmapsto 0} \inf \frac{\mu(B_\epsilon(x)}{\log \epsilon} \geq \rho\} > 0\}$$

The Hausdorff dimension of a measure does not determine alone the possible complicated multiscaling structure of a measure, much more information about the fine scale geometry of a measure is provided by the dimension spectrum (see [3],[28], [29]).

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[1] Supported by "DFG-Schwerpunktprogramm - Dynamik: Analysis, effiziente Simulation und Ergodentheorie".

[2] Our general reference for dimension theory and especially the definition of the Hausdorff dimension of a set ($\dim_H B$) are the books of Falconer [7] and Pesin [28]

Definition 1.2 For $\alpha \geq 0$ let

$$F_\mu(\alpha) = \{x \in X \mid \lim_{\epsilon \longmapsto 0} \frac{\mu(B_\epsilon(x))}{\log \epsilon} = \alpha\}$$

The function $f_\mu(\alpha) = \dim_H F_\alpha$ is called the **dimension spectrum** of μ

Another important spectrum is the Hentschel-Proccacia spectrum which proved to be most accessible to experience and computer simulations (see [28],[12] and [13]).

Definition 1.3 For $\mu \in M(X)$ and $q \geq 0$ (except $q = 1$) we define the **Hentschel-Proccacia spectrum (HP spectrum)** by

$$HP_\mu(q) = \frac{1}{1-q} \lim_{\epsilon \longmapsto 0} \frac{\log \inf_{\mathbf{B}_\epsilon} \sum_{B \in \mathbf{B}_\epsilon} \mu(B)^q}{\log \epsilon}$$

provided that the limit exists. Here the infimum is taken over all countable coverings $\mathbf{B}_\epsilon$ with balls of radius ϵ in X.

Remark 1.1 The value of the HP spectrum it not changed if we take the infimum over all ϵ-grids $\mathbf{G}_\epsilon$ of K where a ϵ-grid is a partition of K into Borel sets such that there is a constant c with $\forall G \in \mathbf{G}_\epsilon \exists x \in G : B_{C\epsilon}(x) \subseteq G \subseteq B_\epsilon(x)$ (see [11]). This means that the HP spectrum is identical with the **Renyi spectrum** introduced in [35].

Remark 1.2 If the measure in question is diametrically regular, i.e. $\exists A > 1, K > 0, r_0 > 0 \forall \epsilon < \epsilon_0 \forall x \in X : \mu(B_{A\epsilon}(x)) \leq K\mu(B_\epsilon(x))$, then in fact it is enough to consider one fixed family of grids $(\mathbf{G}_\epsilon)_{\epsilon > 0}$ in order to calculate the HP spectrum. This makes numerical or experimental calculations possible (see [29])

In the **Multifractal analysis** we are interested in determining $\dim_H \mu, f_\mu(\alpha)$ and $HP_\mu(p)$, studying the analytical properties of the function and relating the quantities to each other. There is one main conjecture in this theory (see [28] and [13]).

Conjecture 1.1 *For "good" measures* $\mu \in M(X)$ *we have that the functions* $(1-q)H_\mu(q)$ *and* $f(\alpha)$ *are analytical, convex and form a Legendre transform pair. Moreover* $\lim_{q \longmapsto 1} H_\mu(q) = \dim_H \mu$.

We know that this conjecture holds for Gibbs measures on conformal repellers and Basic sets for conformal diffeomorphisms. Moreover it holds for Gibbs measures on conformal geometrical constructions (iterated function systems). In all these cases we are able to calculate the involved quantities. The theory that provides such results is exposed in the book of Pesin [28] (see also [29], [30] and [34]). It is based on the thermodynamic formalism and general concepts of dimension theory. Also we like to mention that there are some general results on products and intersection of multifractal measures (see [25])

But for more general classes of measures that are not Gibbs states under conformal processes there is no complete dimension theory and multifractal analysis this days. Only special classes of measures have been analysed and there remain a lot of open problems. We will consider in our work cascade measures which are in general neither Gibbs states nor induced by conformal processes.

2 Deterministic cascade measures

For k Let $T_1, \ldots, T_k$ be contractions of $[0,1]^j$ such that $\{T_1([0,1]^j), \ldots, T_k([0,1]^j)\}$ is a partition of $[0,1]^j$ (the elements cover the set and intersect only in their boundaries) and let $p = (p_1, \ldots, p_k)$ be a probability vector. Define a operator from $M := M([-1,1]^j)$ to M by

$$T\mu = \sum_{i=1}^{k} p_i(\mu \circ T_i^{-1})$$

Theorem 2.1 *There is a unique $\mu \in M$ such that $T\mu = \mu$. Moreover $T^n\ell$ (where ℓ is the normalised Lebesgue measure) converges to μ with respect to the weak* topology*

Remark 2.1 The measure μ may also be described as the image of the Bernoulli measure b^p on $\Sigma := \{1, \ldots, k\}^{I\!N}$ under the coding map $\pi : \Sigma \longmapsto [0,1]^j$ (i.e. $\mu = b^p \circ \pi^{-1}$) given by

$$\pi((s_k)) = \lim_{n \longmapsto \infty} T_{s_1} \circ \ldots \circ T_{s_n}([0,1]^j)$$

Theorem 2.1 can be proved by methods of Hutchinson [15]. We call a measure given by the last proposition **deterministic cascade measures**. We think that it is obvious that such measures may have a quite complicated multiscaling and fractal geometry which is in general hard to describe. In the following the maps T_k are always assumed to be linear. This is of course the simplest case but nevertheless it provides interesting open problems. Moreover one would expect that some results found in the linear case can be generalised later on to case of differentiable maps.

2.1 Self similar deterministic cascade measures

Let us here first discuss a special case which is fairly easy to analyse be know techniques.

Divide $[0,1]$ into k closed subintervals $I_1, \ldots I_k$ of length given by the vector $\beta = (\beta_1, \ldots, \beta_k)$ and let $p = (p_1, \ldots, p_k)$ by a probability vector. By Theorem 2.1 we know that there is a unique $\mu_{\beta,p} \in M([-1,1])$ such that

$$\mu_{\beta,p} = \sum_{i=1}^{k} p_i(\mu_{\beta,p} \circ L_i^{-1})$$

where L_i is the linear function that maps $[0,1]$ to I_i. We call such a measure a **self similar deterministic cascade measure**.

The following theorem gives the complete multifractal analysis of the measure $\mu_{\beta,p}$.

Theorem 2.2 *We have* $HP_{\mu_{\beta,p}}q) = \tau(q)/(q-1)$ *where* $\tau(q)$ *is the solution of*

$$\sum_{i=1}^{k} \frac{p_i^q}{\beta_i^{\tau(q)}} = 1$$

Furthermore Conjecture 1.1 holds for the measure $\mu_{\beta,p}$ *and especially*

$$\dim_H \mu_{\beta,p} = \frac{\sum_{i=1}^{k} p_i \log p_i}{\sum_{i=1}^{k} p_i \log \beta_i}$$

A heuristic argument for this theorem is provided in [13]. A rigours proof in a special situation can be found in[7]. This prove can be easily generalise to get Theorem 2.1 .

We present this theorem as an example for the results one would aim at in more complicated situations.

2.2 Self affine deterministic cascade measures

Divide the square $[0,1]^2$ into rectangles R_i $i = 1, \ldots k$. Furthermore let $p = (p_1, \ldots, p_k)$ be a probability vector. By Theorem 2.1 there is a unique measure $\mu \in M([-1,1])$ such that

$$\mu = \sum_{i=1}^{k} p_i(\mu \circ A_i^{-1})$$

where A_i is the affine function that maps $[0,1]^2$ to R_i. We call such a measure **self affine deterministic cascade measures**. Of course higher dimensional generalisations are possible.

In general not even the dimension of the measures μ is known and it was shown in [21] and [23] that even number theoretical peculiarities can have an influence on these quantity. We only want two discuss here two special cases.

First consider a partition of $[-1,1]^2$ into aligned rectangles $R_{i,j}$ $i = 1, \ldots, h$ $j = 1, \ldots, v$ of equal length $1/h$ and equal weights $1/v$. Furthermore let $P = (p_{i,j})$ be a probability matrix. Denote the corresponding self affine cascade measure by $\mu_{h,v,P}$. The Hausdorff dimension of this measures was found in [20], it may also be calculated using the general dimension theory of ergodic measures (see [16] and [19])

Theorem 2.3 *We have*

$$\dim_H \mu_{h,v,P} = \frac{1}{h} \sum_{i,j} p_{i,j} \log p_{i,j} + \left(\frac{1}{h} - \frac{1}{v}\right) \sum_j \left(\sum_i p_{i,j} \log\left(\sum_i p_{i,j}\right)\right)$$

On the other hand as far as we know there is no multifractal analysis for this measures. Thus we have

Problem 2.1 Calculate $f_{\mu_{h,v,P}}(\alpha)$ and $H_{\mu_{h,v,P}}(q)$. Does Conjecture 1.1 hold?

In this special situation we have hope that we are able to solve the problem using the special projection properties of the measure at hand.

Now consider a very special class of self-affine cascade measures which is more paradigmatic in view of the general case because there are overlaps in the projections. Choose a rectangle R_1 of height τ_1 and width β_1 in the upper right corner of the square and assign it the probability p. . Choose a rectangle R_2 of height τ_2 and width β_2 in the lower left corner of the square and assign it the probability $1-p$. All other rectangles of the covering we can choose arbitrary with probability $1-p$. Assume that $\beta_1+\beta_2>1$ and $\tau_1+\tau_2<1$.For $\vartheta=(\beta_1,\beta_2,\tau_1,\tau_2,p)$ be μ_ϑ br the corresponding self affine cascade measure.

Theorem 2.4 *We have*

$$\dim_H \mu_\vartheta = \frac{p\log p+(1-p)\log(1-p)}{p\log\tau_1+(1-p)\log\tau_2}+(1-\frac{p\log\beta_1+(1-p)\log\beta_2}{p\log\tau_1+(1-p)\log\tau_2})\dim pr\mu_\vartheta$$

where pr is the projection on the first coordinate axis. Furthermore generically in the sense of Lebesgue measure on the parameter space we have $\dim_H pr\mu_\vartheta=1$ *if* $(\beta_2 p)^p(\beta_1(1-p))^{1-p}\le\beta_1\beta_2$.

The proof of this result can be found in [22]. The first part of the theorem uses the general dimension theory for ergodic measures (see [19], [2]) and the second part uses a general approach in geometric measure theory(see [18], [26], [27]). The strategy used in the proof seems to be useful in great generality. Thus we have hope to solve the following problem

Problem 2.2 Find $\dim_H \mu$ for large classes of deterministic self-affine cascade measure using the strategies developed in the proof of 2.4 .

Of course the next step would be solve the following problem:

Problem 2.3 What are the multifractal properties of the measures μ_ϑ and of more general classes of self-affine cascade measures.

We think about this problem but we are at this stage of development not sure if will be able solve it. There is one approach to find the dimension (see [8]) and estimates on the multifractal spectra (see [33]) of self-affine measures for almost all translations if the contraction rates of the affine maps are less than 1/3. But the multifractal analysis in this situation is not complete. Moreover the techniques do not apply to the measures μ_ϑ and other simple classes of self-affine cascade measures. We hope that a multifractal analysis of self similar measures with overlaps may help to get a little bit nearer to a solution of problem 2.3 .

3 Random cascade measures

For k let $T_1, \ldots, T_k$ be again contractions of $[0,1]^j$ such that $\{T_1([0,1]^j), \ldots, T_k([0,1]^j)\}$ is a partition of $[0,1]^j$ and let $\pi : \Sigma \longmapsto [0,1]^j$ be the map defined in Remark 2.1 . Furthermore for all finite sequences $(s_1, \ldots, s_n) \in \{1, \ldots, k\}^n$ let $W(s_1, \ldots, s_n)$ be i.i.d nonnegative mean one random variables and let W be the common distribution of all of this variables. Define a sequence b_n of discrete random measures on Σ by

$$b_n([s_1, \ldots, s_n]) = W(s_1)W(s_1, s_2) \ldots W(s_1, \ldots s_n)$$

where $[s_1, \ldots, s_n]$ denotes a cylinder set in Σ. Consider the sequence of random measures $\mu_n = b_n \circ \mu^{-1}$ on $[0,1]^v$. By means of martingales it can be shown that:

Theorem 3.1 *The sequence of random measures μ_n converges almost surly to a limit μ with respect to the weak* topology.*

We call a measure μ of the type given by Theorem 3.1 **random cascade measure**. The construction described here is a generalisation of the construction done in [14]. A axiomatic approach to random fractal measures can be found in [1].

In analogy to 2.1 we can define **self similar random cascade measures** $\mu_{\beta,W}$. In the special case $\beta = (1/k, \ldots, 1/k)$ we set $\mu_{k,W} := \mu_{\beta,W}$. In the case that W is strongly bounded. the multifractal analysis of this measures was developed in [14].

Theorem 3.2 *Let W be strongly bounded i.e. $\exists a > 0$ such that $P(W \geq a) = 1$ and $P(W < k) = 1$. Assume that $EW^{2h}/(EW^h)^2 < k$ where E denotes the expectation. Then with probability one we have*

$$HP_{\mu_{k,W}}(q) = \frac{\log EW^q}{(q-1)\log k} - 1$$

and conjecture 1.1 holds.

In view of this theorem we naturally have the following problem

Problem 3.1 Generalise Theorem 3.1 to all self-similar random cascade measures $\mu_{\beta,W}$.

In analogy to 2.2 we are able to define **self-affine random cascade measures**. We to restrict our attention to the first case described in 2.1 (aligned rectangles). In analogy to the deterministic measure $\mu_{h,v,P}$ we define stochastic measure $\mu_{h,v,W}$. Obviously we have the following problem.

Problem 3.2 What are the multifractal properties of the measures $h_{h,v,W}$ almost surely?

If we are able to solve Problem 2.1 we have some hope to generalise our techniques from the deterministic to the random case in order to solve this problem.

4 Relations to cascade and multifractal models in the theory of turbulence

It is well known that a complete analytical treatment of the Navier-Stokes equitation describing phenomena in fluid mechanics is still far out of reach. Thus simpler mathematical models were invented to study the interesting praenomen of turbulence in fluid mechanics. Especially **cascade models** like the Gyroscope model seems to be able to describe some important features of turbulence.(see [31] and references in this article)
Mandelbrot [17] observed that the energy dissipation from large scale to smaller scale in cascade models induces (for scale to zero) an energy dissipation which is in general concentrated on a fractal set. This lead to the development of the so called **multifractal models** of turbulence that describe the transfer of energy over the intiarl range. In this models energy distribution on one structure(eddy) is mapped to the energy dissipation of substructures(eddys) on smaller scales. As the result one gets a cascade measure with a possible complicated multifractal structure.
Frisch and Parisi developed in analogy to Kolmogorov an axiomatic approach which assumes multifractal structures in turbulence qua hypothesis (see [9], [7]). Stochastic cascade models of turbulence were for introduced in [24] and [36] A explicit multifractal cascade model of turbulence is described in [31]. The description of a deterministic cascade model can be found in [6].

There is one general problem, concerning this multifractal models of turbulence which should be looked at more carefully from the analytical point of view then it was done in the literature we know:

Problem 4.1 To which extend do multifractal models and especially cascade measure in fact describe the stochastic properties of cascade models of turbulence?

The second important problem is concerned with the use of the multifractal formalism in the physical literature on turbulence. Often we find only heuristic arguments and numerical calculations. Mathematical statements like Conjecture 1.1 are frequently used also they are not rigour proofed.

Problem 4.2 Find exact mathematical descriptions of the constructions used in multifractal models of turbulence and justify the use of the multifractal formalism in the theory of turbulence rigoursly.

The formal description of cascade measures done in the section two is of course one step to solve this problem. The study of self-affine structures in 2.1 is of special interest because in modern multifractal models of turbulence one assumes generalised scale invariance which leads in the case of linear rescaling at different rates in different direction to self affine cascade measures (see section two in [31]. The generalisation from the deterministic to random cascade measures we pointed at in 2.2 is of the same importants because multifractal models of turbulence invented by physicist are usually described stochastically.

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