Dimension Theory of Linear Solenoids

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Abstract

We develop the dimension theory for a class of lineare solenoids, which have a "fractal" attractor. We will find the dimension of the attractor, proof formulas for the dimension of ergodic measures on this attractor and discuss the question whether there exists a measure of full dimension.
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1   Introduction

We consider in this article a class of dynamical systems given by piece-wise lineare maps acting on a cube. These dynamical system are very similar to the classical Smale-Williams Solenoids (see [6]), having a one dimensional unstable and a two dimensional stable manifold. As the solenoid the systems we study have a global attractor, which has a complicated "fractal" geometry. Thus we will discuss here the "fractal" dimension of this attractor. In Theorem 3.1 we determine the Hausdorff and Box-counting dimension, using results on the dimension of self-affine sets in the plane found in [9]. In the following we will apply symbolic dynamics and the theory of hyperbolic dynamics to the class of lineare solenoids. In section four we will find a coding of the dynamics through a shift on two symbols on a set of full measure. This allows us to to find a representation of all ergodic measures for the systems as images of shift ergodic measures under the coding map. In section fife we will demonstrate the existence of Lyapunov exponents and Lyapunov charts for lineare solenoids with respect to any ergodic measure. This is the background we need to apply the general dimension theory of hyperbolic measures, see [7] and [2]. Using this theory we will show that ergodic measure for our systems are exact dimensional. Moreover we will find a formula for the dimension of ergodic measures in terms of entropy and Lyapunov exponents and the dimension of transversal measures (see Theorem 6.1). For Bernoulli measures this formula yield an explicit expression (see Corollary 6.2) for the dimension in terms of self-similar measure studied in [9] . In the last section of this article we will discuss the question whether there exists an ergodic measure of full dimension, which means that the dimension of the ergodic measure equals the dimension of the attractor. This question is widely open in the dimension theory of dynamical systems. It is of particular interest since ergodic measures of full dimension are of great geometrical significants, describing the long term behavior of orbits on the whole attractor in the dimensionaltheoretical sense . Results of Manning and McClusky [8] show that in the case of horseshoes diffeomorphisms there does not exist an ergodic measure of full dimension in general. One can not maximize the dimension in the stable and in the unstable direction at the same time. In [10] we demonstrate that for generalized Baker's transformations there exists parameter domains for which a measure of full

dimension exists and parameter domains where the dimension of the invariant set can not even be approximated by the dimension of ergodic measures. We observe the same phaenomenon in the case of lineare solenoids. We will show that there are manifolds in the parameter domain where there is a measure of full dimension and manifold where the variational principle of dimension does not hold (see Theorem 7.1). At the end of this paper the reader, who is not familiar with dimension theory, will find an appendix containing a short introduction to this field.

Acknowledgment

2 Linear solenoids

Let $\mathbb{W} = [-1, 1]^3$. We consider the following class of piecewise affine maps $f_v : \mathbb{W} \longmapsto \mathbb{W}$ given by

$$f_v(x, y, z) = \{ \begin{array}{ll} (2x - 1, \beta_1 y + (1 - \beta_1), \tau_1 z + (1 - \tau_1)) & \text{if} \quad x \geq 0 \\ (2x + 1, \beta_2 y - (1 - \beta_2), \tau_2 z - (1 - \tau_2)) & \text{if} \quad x < 0 \end{array}$$

where we assume

$$v = (\beta_1, \beta_2, \tau_1, \tau_2) \in (0, 1)^4 \quad \text{and} \quad \tau_1 + \tau_2 < 1.$$
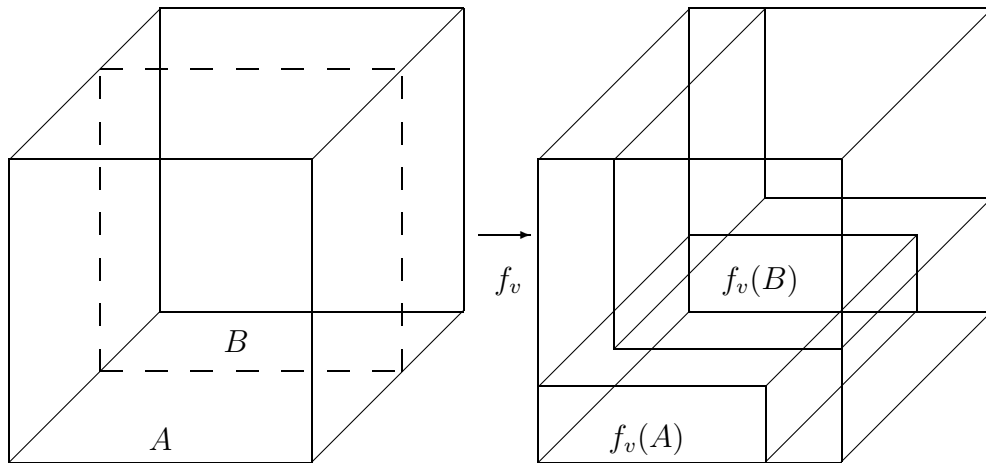

Figure 1: The action of f_v on the cube $\mathbb{W}$.

Obvoiusly the maps f_v are invertible and there is a global Attraktor for the maps given by

$$\Lambda_v = \text{closure}(\bigcap_{k=0}^{\infty} f_v^n(\mathbb{W})).$$

We call the system (Λ_v, f_v) a linear Solenoid. We see that this system is quite similar to the classical Smale-Williams Solenoid $(\Delta_{\beta,\tau}, g_{\beta,\tau})$, see [6]. The Smale-Williams Solenoid is constructed by a family of maps $g_{\beta,\tau} : \mathbb{T}^2 \longmapsto \mathbb{T}^2$ on the full torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{D}^2$ defined by

$$g(\phi, y, z) = (2\phi \bmod 2\pi, \beta y + 1/2\cos(2\pi\phi), \tau z + 1/2\sin(2\pi\phi))$$

$$\text{with} \qquad \beta, \tau \in (0, 1/2).$$

$g_{\beta,\tau}$ has the global attractor

$$\Delta_{\beta,\tau} = \bigcap_{n=0}^{\infty} g_{\beta,\tau}^n(\mathbb{T}^2).$$
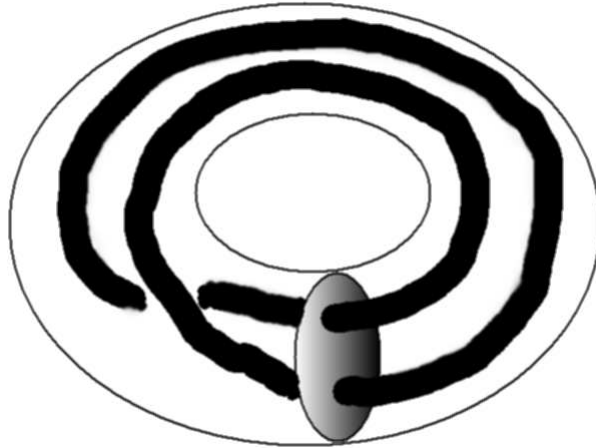

Figure 2 The action of $g_{\beta,\tau}$ on the full Torus $\mathbb{T}^2$.

In fact the systems $(\Delta_{\beta,\tau}, g_{\beta,\tau})$ and (Λ_v, f_v) have similar properties. Both systems are expanding in the first coordinate direction with expansion rate $\log 2$ and contracting in the two other coordinate direction. Moreover both maps are invertible and into with a global attractor. The classical Solenoid is hyperbolic and conjugated to the full Shift on two symbols, see [6]. In section four we will show that our linear Solenoids are up to a set of measure zero as well hyperbolic and conjugated to the full shift on two symbols. The last similarity is that both Λ_v and $\Delta_{\beta,\tau}$ have a complicated non smooth geometry. Dimensional theoretical properties of the classical Solenoid where extensity studied, see [1],[16] or [12]. We will develop here the dimensiontheory for lineare solenoids. In the next chapter we will present our results on the dimension of the attractor Λ_v.

3 Dimension of the attractor

We first give here an simple description of the attractor Λ_v using iterated function systems, see [4].

Proposition 3.1 *We have*

$$\Lambda_v = [-1, 1] \times \Lambda_v^s$$

where Λ_v^s is the unique compact set fulfilling

$$\Lambda_v^s = T_{\beta_1, \tau_1}(\Lambda_v^s) \cup T_{\beta_2, \tau_2}(\Lambda_v^s)$$

with $T_{\beta_1, \tau_1}, T_{\beta_2, \tau_2} : [-1, 1]^2 \longmapsto [-1, 1]^2$ given by

$$T_{\beta_1, \tau_1}(y, z) = (\beta_1 y + (1 - \beta_1), \tau_1 z + (1 - \tau_1))$$

$$T_{\beta_2, \tau_2}(y, z) = (\beta_2 y + (1 - \beta_2), \tau_2 z + (1 - \tau_2)).$$

Proof. Let $T_1 := T_{\beta_1, \tau_1}$ and $T_2 = T_{\beta_2, \tau_2}$. We have

$$\text{closure}(f_v(\mathbb{W})) = [-1, 1] \times T_1(\mathbb{W}) \cup [-1, 1] \times T_2(\mathbb{W})$$

and hence

$$\text{closure}(f_v^n(\mathbb{W}) = [-1, 1] \times \bigcup_{s_1, \ldots, s_n \in \{1, 2\}} T_{s_1} \circ T_{s_2} \circ \ldots \circ T_{s_2}(\mathbb{W})$$

Now let

$$\Lambda_v^s = \bigcap_{n=1}^{\infty} \bigcup_{s_1, \ldots, s_n \in \{1, 2\}} T_{s_1} \circ T_{s_2} \circ \ldots \circ T_{s_2}(\mathbb{W})$$

By this definition we get

$$\Lambda_v = [-1, 1] \times \Lambda_v^s.$$

Moreover Λ_v^s is compact with

$$\Lambda_v^s = T_1(\Lambda_v^s) \cup T_2(\Lambda_v^s)$$

Uniqueness of Λ_v^s with this property follows from [5]. $\qquad\square$

Our results on the dimension of the attractor Λ_v is now mainly a consequence of our results on the self-affine sets Λ_v^s given in [9] and and [12] . In the following we denote by $\dim_B A$ the box-Counting dimension and by $\dim_H A$ the Hausdorff dimension of a set A; we refer to the appendix of thsi work for the definition of these qualtities.

Theorem 3.1 *Let* $v = (\beta_1, \beta_2, \tau_1, \tau_2) \in (0, 1)^4$ *with* $\beta_1 + \beta_2 > \tau_1 + \tau_2$.

If $\beta_1 + \beta_2 < 1$ *we have*

$$\dim_B \Lambda_v = \dim_H \Lambda_v = d + 1$$

where d *is the solution of*

$$\beta_1^d + \beta_2^d = 1.$$

If $\beta_1 + \beta_2 \geq 1$ *we have*

$$\dim_B \Lambda_v = d + 2$$

where d *is the solution of*

$$\beta_1 \tau_1^d + \beta_2 \tau_2^d = 1.$$

Moreover for almost all $\beta_1, \beta_2 < 0.649$ *we have*

$$\dim_H \Lambda_v = \dim_B \Lambda_v.$$

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Proof. By proposition 8.1 of the appendix and proposition 3.1 we have

$$\dim_{H/B} \Lambda_v = \dim_{H/B} \Lambda_v^s + 1.$$

If $\beta_1 + \beta_2 < 1$ we have by example 16.3 of [12] $\dim_B \Lambda_v = \dim_H \Lambda_v = d$.

If $\beta_1 + \beta_2 \geq 1$ we get by theorem III of [9] $\dim_B \Lambda_v^s = d + 2$ and generically $\dim_H \Lambda_v^s = \dim_B \Lambda_v^s$ under the assumption that $\beta_1, \beta_2 > 0.649$. $\qquad\square$

The condition $\beta_1, \beta_2 < 0.649$ in the last theorem is due to the technique we used in [9]. We do not belief that this condition is essential, also we were not able to omit it. The identity of Box-Counting and Hausdorff dimension in the last statement does not hold in general. In [10] we described numbertheoretical exceptions in the symmetric case $\beta_1 = \beta_2$.

4 Shift coding of the Dynamics

We need some notation to introduce a symbolic coding of the dynamics of system (Λ_v, f_v).

Let $\Sigma = \{-1, 1\}^{\mathbb{Z}}$ be the Shift space. With the product metric defined by

$$d(s, t) = \sum_{k=-\infty}^{\infty} |s_k - t_k| 2^{-|k|}$$

Σ becomes a perfect, totally disconnected and compact metric space; see [3]. The forward shift map σ on Σ is given by $\sigma((s_k)) = (s_{k+1})$, the backward shift σ^{-1} is given by $\sigma((s_k)) = (s_{k-1})$.

For a sequence $s \in \Sigma$ and $\gamma_1, \gamma_2 \in (0, 1)$ we define a map

$$\hat{\pi}_{\gamma_1, \gamma_2} : \Sigma^+ \longrightarrow [\frac{-\gamma_2}{1 - \gamma_2} \frac{\gamma_1}{1 - \gamma_1}]$$

by

$$\hat{\pi}_{\gamma_1, \gamma_2}(s) = \sum_{k=0}^{\infty} s_k \gamma_2^{\sharp(s,k)} \gamma_1^{\bar{\sharp}(s,k)}$$

where

$$\sharp(s, k) = \mathrm{Cardinality}\{s_i | s_i = -1 \ \ i = 1, \dots, k\}$$
$$\bar{\sharp}(s, k) = \mathrm{Cardinality}\{s_i | s_i = +1 \ \ i = 1, \dots, k\}.$$

Let L_{γ_1, γ_2} be the monoton increasing linear map form $[\frac{-\gamma_2}{1-\gamma_2} \frac{\gamma_1}{1-\gamma_1}]$ onto $[-1, 1]$ and let $\pi_{\gamma_1, \gamma_2} = L_{\gamma_1, \gamma_2} \circ \hat{\pi}_{\gamma_1, \gamma_2}$. Moreover we define the map of the signed dyadic expansion

$$i : \Sigma \longmapsto [-1, 1]$$

by

$$i(s) = \sum_{k=1}^{\infty} s_{-k} (1/2)^k.$$

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For $v = (\beta_1, \beta_2, \tau_1, \tau_2) \in (0,1)^4$ we define the coding map

$$\pi_v : \Sigma \longmapsto \Lambda_v$$

by

$$\pi_v(s) = (i(s), \pi_{\beta_1, \beta_2}(s), \pi_{\tau_1, \tau_2}(s)).$$

Proposition 4.1 π_v *is continuous and onto* Λ_v. *Moreover the map is bijective from*

$$\bar{\Sigma} := \Sigma \backslash \{(s_k) | \exists k_0 \forall k \leq k_0 \in \mathbb{Z} : s_k = 1\} \cup \{(1)\}$$

onto Λ_v *and we have*

$$\forall s \in \bar{\Sigma} \quad : \quad \pi_v(\sigma^{-1}(s)) = f_v(\pi_v(s)).$$

Proof. It is obvious that $\hat{\pi}_{\gamma_1, \gamma_2}$ is continuous since

$$d(s,t) \leq \frac{1}{2^n} \Rightarrow s_k = t_k \text{ for } k = 0, \ldots, n \Rightarrow |\hat{\pi}_{\gamma_1, \gamma_2}(s) - \hat{\pi}_{\gamma_1, \gamma_2}(t)| \leq \frac{\gamma_1^{n+1}}{1 - \gamma_1} + \frac{\gamma_2^{n+1}}{1 - \gamma_2}.$$

Therefor π_v is continuous. By the product structure of the map π_v and $i(\Sigma) = [-1, 1]$ we have $\pi_v(\Sigma) = [-1, 1] \times A$. Furthermore a simple calculation using the maps π_{β_1, β_2} and π_{τ_1, τ_2} shows

$$A = T_{\beta_1, \tau_1}(A) \cup T_{\beta_2, \tau_2}(A).$$

By proposition 3.1 we have $A = \Lambda_v^s$ and thus $\pi_v(\Sigma) = \Lambda$. Since i is invertible on $\bar{\Sigma}$ and $i(\bar{\Sigma}) = [-1, 1]$ the map π_v is bijective from $\bar{\Sigma}$ onto Λ. Just looking at the definition of $\hat{\pi}_{\gamma_1, \gamma_2}$ we see that

$$\hat{\pi}_{\gamma_1, \gamma_2}((s_{k+1})) = \{ \begin{array}{ll} \gamma_1^{-1} \hat{\pi}_{\gamma_1, \gamma_2}((s_k)) - 1 & \text{if} \quad s_0 = +1 \\ \gamma_2^{-1} \hat{\pi}_{\gamma_1, \gamma_2}((s_k)) + 1 & \text{if} \quad s_0 = -1 \end{array}.$$

Hence we have

$$\pi_{\gamma_1, \gamma_2}((s_{k+1})) = L_{\gamma_1, \gamma_2}(\hat{\pi}_{\gamma_1, \gamma_2}((s_{k+1}))) = \{ \begin{array}{ll} L_{\gamma_1, \gamma_2}(\gamma_1^{-1} \hat{\pi}_{\gamma_1, \gamma_2}((s_k)) - 1) & \text{if} \quad s_0 = 1 \\ L_{\gamma_1, \gamma_2}(\gamma_2^{-1} \hat{\pi}_{\gamma_1, \gamma_2}((s_k)) + 1) & \text{if} \quad s_0 = -1 \end{array}$$

$$= \{ \begin{array}{ll} \gamma_1^{-1} \pi_{\gamma_1, \gamma_2}((s_k)) + (1 - \gamma_1^{-1}) & \text{if} \quad s_0 = 1 \\ \gamma_2^{-1} \pi_{\gamma_1, \gamma_2}((s_k)) - (1 - \gamma_2^{-1}) & \text{if} \quad s_0 = -1 \end{array}.$$

For $s \in \bar{\Sigma}$ we have

$$i((s_{k+1})) = \sum_{k=1}^{\infty} s_{-k+1} 2^{-k} \geq 0 \Leftrightarrow s_0 = +1.$$

Thus

$$f_v \circ \pi_v((s_{k+1})) =$$

$$\{ \begin{array}{l} (2i((s_{k+1})) - 1, \beta_1 \pi_{\beta_1, \beta_2}((s_{k+1})) + (1 - \beta_1), \tau_1 \pi_{\tau_1, \tau_2}((s_{k+1})) + (1 - \tau_1)) \text{ if } s_0 = +1 \\ (2i((s_{k+1})) + 1, \beta_2 \pi_{\beta_1, \beta_2}((s_{k+1})) - (1 - \beta_2), \tau_2 \pi_{\tau_1, \tau_2}((s_{k+1})) - (1 - \tau_2)) \text{ if } s_0 = -1 \end{array}.$$

$$= (i((s_k)), \pi_{\beta_1, \beta_2}((s_k)), \pi_{\tau_1, \tau_2}((s_k))) = \pi_v((s_k)).$$

σ as a map of Σ is invertible and we get $f_v(\pi_v(s)) = \pi_v(\sigma^{-1}(s))$. $\qquad\square$

We can now represent all ergodic measures of the system (Λ_v, f_v) using the coding map π. Again we need some notations. Given a compact metric space X we denote the set of all Borel probability measures on X by $M(X)$. With the weak* topology $M(X)$ becomes a compact, convex and metricable space. If T is a Borel measurable transformation on X we call a measure μ T-invariant if

$$T(\mu) := \mu \circ T^{-1} = \mu.$$

The set of all invariant measures forms a compact, convex and nonempty subset of $M(X)$. An invariant measure μ is called ergodic if

$$T^{-1}(B) := B \Rightarrow \mu(B) \in \{0, 1\}$$

hold for all Borel subsets B of the space X. The set of all ergodic measures

$$M(X, T) := \{\mu \in M(X) | \mu \ \ T\text{-ergodic}\}$$

is nonempty, convex and compact with respect to the weak* topology. It consists of the extreme points of the set of invariant measures. By b^p for $p \in (0, 1)$ we denote the Bernoulli measure on Σ, which is the product of the discrete measure giving 1 the probability p and -1 the probability $(1-p)$. The Bernoulli measures are ergodic with respect to forward and backward shifts. Given b^p on $\{-1, 1\}^{\mathbb{Z}^-}$ we define the corresponding Bernoulli measure ℓ^p on $[-1, 1]$ by $\ell^p = i(b^p)$. For the basic facts in ergodic theory mentioned here we refer to [3], [17] or [6].

Proposition 4.2 *The map*
$$\mu \longmapsto \mu_\vartheta := \mu \circ \pi_\vartheta^{-1}$$
is a affine homeomorphism from $M(\Sigma, \sigma)$ onto $M(\Lambda_\vartheta, f_\vartheta)$. Moreover b_ϑ^p is a product of the Bernoulli measure on Λ_v^s with ℓ^p.

Proof. It follows directly form proposition 4.1 and proposition 3.1 of [3] that the map is continuous, affine and surjective. We will use the fact that π_ϑ restricted to $\bar{\Sigma}$ is a bijection to show that the map is invertible. Given a measure $\xi \in M(\Lambda_\vartheta, f_\vartheta)$ we define a measure $F\xi \in M(\Sigma)$ by $F\xi(B) = \xi(\pi_\vartheta(B))$ for all Borel sets in Σ. Let μ be in $M(\Sigma, \sigma)$ and B be a Borel set in Σ. We have $F\mu_v(B) = \mu(\pi_v^{-1}(\pi_v(B))) = \mu(\bar{\Sigma} \cap \pi_v^{-1}(\pi_v(B))) = \mu(\bar{\Sigma} \cap B) = \mu(B)$. Hence F is the inverse map to $\mu \longrightarrow \mu_\vartheta$. The continuity of F follows by the compactness of $M(\Sigma, \sigma)$. The product structure of b_v^p is just an obvious consequence of the product structure of the map π_v. $\qquad\square$

5   Hyperbolicity

We will show here that there exists expansion and contraction rates (Lyapunov exponents) on the Solenoid (Λ_v, f_v) for a set of full measure with respect to any ergodic measure $\mu \in M(\Lambda_v, f_v)$.

Lemma 5.1 *There is a subset $\Omega_v \subseteq \Lambda_v$ which has full measure for all $\mu_v \in M(\Lambda_v, f_v)$ such that f_v is a bijection on Ω_v and f_v is differentiable for all $\mathbf{x} \in \Omega_v$ with*

$$D_{\mathbf{x}} f_v = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \tau_{\tau_1} \end{pmatrix} \quad \textit{if} \quad y > 0 \quad \textit{and} \quad D_{\mathbf{x}} f_v = \begin{pmatrix} \beta_2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \tau_2 \end{pmatrix} \quad \textit{if} \quad y < 0.$$

Proof. Denote the singularity $\{0\} \times [-1,1]^2$ of the map f_v by S and define the set Ω_v by

$$\Omega_v = \bigcap_{n=-\infty}^{\infty} f_v^n(\mathbb{W} \backslash S).$$

By definition we have $f_v(\Omega_v) = \Omega_v$ and since f_ϑ is injective it is in fact a bijection on Ω_v. Moreover if $(x, y, z) \in \Omega_v$ then $(x, y, z) \notin S$ and hence f_v is differentiable and has obviously the derivative that we stated in the lemma. It remains to show that $\mu_v(\Omega_v) = 1$. By elemental calculations we see that

$$\Omega_v = (\{(x, y, z) \in \Lambda_v | y \neq 1, \quad y \neq -1\} \cup \{(1,1,1), (-1,-1,-1)\}) \backslash \bigcup_{n=0}^{\infty} f^{-n}(S).$$

Since μ_v is invariant and the union in the expression above is disjoint it has zero measure. It remains to show that $\mu_v(\{1\} \times [-1,1] \times [-1,1]) = \mu_v(\{(1,1,1)\}$ and $\mu_v(\{-1\} \times [-1,1] \times [-1,1]) = \mu_v(\{(-1,-1,-1)\})$. But this is obvious since f_v is just a contraction with fixed point $(1,1,1)$ resp. $(-1,-1,-1)$ on the sets $\{1\} \times [-1,1] \times [-1,1]$ resp. $\times \{1\} \times [-1,1] \times [-1,1]$. $\qquad\square$

We now define linear subspaces of $\mathbb{W}$ by

$$\mathbb{E}^u = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \quad \mathbb{E}^s = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$$\mathbb{E}^{ss} = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \quad \mathbb{E}^{ws} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle .$$

Given a Borel measure μ on Σ and $\gamma_1, \gamma_2 \in (0,1)$ we write

$$\Xi_{\gamma_1, \gamma_2}^{\mu} = \mu(\{s \in \Sigma | s_0 = 1\}) \log \gamma_1 + \mu(\{s \in \Sigma | s_0 = -1\}) \log \gamma_2.$$

Proposition 5.1 *Given $\mu \in M(\Sigma, \sigma)$ we have for μ_v-almost all $\mathbf{x} \in \Lambda_v$*

$$\lim_{n \longrightarrow \infty} \frac{1}{n} \log \|D_{\mathbf{x}} f_\vartheta^n \overrightarrow{v}\| = \log 2 \qquad \forall \overrightarrow{v} \in \mathbb{E}^u$$

$$\textit{If} \quad \Xi_{\beta_1, \beta_2}^{\mu} \geq \Xi_{\tau_1, \tau_2}^{\mu} \quad : \quad \lim_{n \longrightarrow \infty} \frac{1}{n} \log \|D_{\mathbf{x}} f_\vartheta^n \overrightarrow{v}\| = \left\{ \begin{array}{ll} \Xi_{\beta_1, \beta_2}^{\mu} & \textit{if} \quad \overrightarrow{v} \in \mathbb{E}^s \backslash \mathbb{E}^{ss} \\ \Xi_{\tau_1, \tau_2}^{\mu} & \textit{if} \quad \overrightarrow{v} \in \mathbb{E}^{ss} \end{array} \right.$$

$$\textit{If} \quad \Xi_{\beta_1, \beta_2}^{\mu} \leq \Xi_{\tau_1, \tau_2}^{\mu} \quad : \quad \lim_{n \longrightarrow \infty} \frac{1}{n} \log \|D_{\mathbf{x}} f_\vartheta^n \overrightarrow{v}\| = \left\{ \begin{array}{ll} \Xi_{\tau_1, \tau_2}^{\mu} & \textit{if} \quad \overrightarrow{v} \in \mathbb{E}^s \backslash \mathbb{E}^{ws} \\ \Xi_{\beta_1, \beta_2}^{\mu} & \textit{if} \quad \overrightarrow{v} \in \mathbb{E}^{ws} \end{array} \right.$$

Proof. By lemma 5.1 we have for μ_v-almost all $\mathbf{x} \in \Lambda_v$ and all $n > 0$

$$\log ||D_{\mathbf{x}} f_v^n (\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix})|| = n \log 2 + \log y.$$

This implies our claim about $\mathbb{E}^u$. Now consider $\mathbb{E}^s$. By lemma 5.1, proposition 4.1 and 4.2 we have for μ_v-almost all $\mathbf{x} \in \Lambda_v$ and all $n > 0$

$$\log ||D_{\mathbf{x}} f_v^n (\begin{pmatrix} x \\ 0 \\ z \end{pmatrix})|| = \log \sqrt{(x\beta_1^{n - \bar{\sharp}_n(s) + 1} \beta_2^{\bar{\sharp}_n(s)})^2 + (z\tau_1^{n - \bar{\sharp}_n(s) + 1} \tau_2^{\bar{\sharp}_n(s)})^2}$$

where $s = (s_k) = \pi_v^{-1}(\mathbf{x})$ and $\bar{\sharp}_n(s)$ counts the number of entries in the set $\{s_0, s_{-1}, \ldots, s_{-n}\}$ that are -1. We now have to determine the limit of this expression for μ- almost all $s \in \Sigma$. By Birkhoffs ergodic theorem (see 4.1.2 of [6]) we have:

$$\lim_{n \longrightarrow \infty} \frac{1}{n+1} \sum_{k=0}^{n} f(\sigma^{-k}(s)) = \int f d\mu$$

for almost all $s \in \Sigma$ and all L^1 functions f on Σ with respect to μ. Applying this to the functions

$$f_{ws}(s) = \{ \begin{matrix} \log \beta_1 & \text{if} & s_0 = 1 \\ \log \beta_2 & \text{if} & s_0 = -1 \end{matrix} \qquad f_{ss}(s) = \{ \begin{matrix} \log \tau_1 & \text{if} & s_0 = 1 \\ \log \tau_2 & \text{if} & s_0 = -1 \end{matrix}$$

we obtain

$$\lim_{n \longrightarrow \infty} \frac{1}{n} \log \beta_1^{n - \bar{\sharp}_n(s) + 1} \beta_2^{\bar{\sharp}_n(s)} = \Xi_{\beta_1, \beta_2}^{\mu}$$

and

$$\lim_{n \longrightarrow \infty} \frac{1}{n} \log \tau_1^{n - \bar{\sharp}_n(s) + 1} \tau_2^{\bar{\sharp}_n(s)} = \Xi_{\tau_1, \tau_2}^{\mu}$$

almost everywhere and from this by elemental calculus

$$\lim_{n \longrightarrow \infty} \frac{1}{n} \log \sqrt{(x\beta_1^{n - \bar{\sharp}_n(s) + 1} \beta_2^{\bar{\sharp}_n(s)})^2 + (z\tau_1^{n - \bar{\sharp}_n(s) + 1} \tau_2^{\bar{\sharp}_n(s)})^2} = \max\{\Xi_{\beta_1, \beta_2}^{\mu}, \Xi_{\tau_1, \tau_2}^{\mu}\}$$

if $x \neq 0$ and $y \neq 0$. This implies our claims about the stable directions.

$\square$

Proposition 5.1 means that Lyapunov exponents exists almost everywhere for the ergodic systems $(\Lambda_v, f_v, \mu_\vartheta)$. $\mathbb{E}^u$ is the unstable direction with Lyapunov exponent $\log 2$ and $\mathbb{E}^s$ is the stable direction with exponent $\Xi_{\beta_1, \beta_2}^{\mu}$ or $\Xi_{\tau_1, \tau_2}^{\mu}$ depending on which quantity is bigger. Accordingly $\mathbb{E}^{ss}$ (resp. E^{ws}) is the strong stable direction with Lyapunov exponent $\Xi_{\tau_1, \tau_2}^{\mu}$ (resp. $\Xi_{\beta_1, \beta_2}^{\mu}$). In order to guarantee the existence of Lyapunov charts associated with the Lyapunov exponents we have to show that the set of points that does not approach the singularity $S = \times\{0\} \times [-1, 1] \times [-1, 1]$ with exponential rate has full measure, see [14]. Precisely we have:

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Proposition 5.2 *Given $\mu \in M(\Lambda_v, f_v)$ we have for all $\epsilon > 0$*

$$\mu_v(\{\mathbf{x} \in \Lambda_v | \exists l > 0 \; \forall n > 0 \; : \; d(f^n(\mathbf{x}), S) > (1/l)e^{-\epsilon n}\}) = 1.$$

Proof. Fix $\epsilon > 0$. First note that it is sufficient if we show

$$\mu_\vartheta(\{\mathbf{x} \in \Lambda_\vartheta | \exists (n_k)_{k \in \mathbb{N}} \longrightarrow \infty \; \forall k > 0 \; : \; d(f^{n_k}(\mathbf{x}), S) \leq e^{-\epsilon n_k}\}) = 0$$

because if we have for a point $\mathbf{x}$ that $\exists n_0 \forall n > n_0 \; d(f^n(\mathbf{x}), S) > e^{-\epsilon n}$ then there exists $l > 0$ such that $d(f^n(\mathbf{x}), S) > (1/l)e^{-\epsilon n} \; \forall n > 0$.

By proposition 4.2 and the definition of the measure μ_v this assertion is equivalent to $\mu(N) = 1$ where

$$N := \{s \in \hat{\Sigma} | \exists (n_k)_{k \in \mathbb{N}} \longrightarrow \infty \; \forall k > 0 \; d(\sigma^{-n_k}(s), \tilde{S}) \leq e^{-\epsilon n_k}\}$$

and $\tilde{S}$ is the singularity in the symbolic coding, i.e.

$$\tilde{S} = \{s \in \Sigma | s_{-1} = 1 \quad s_k = -1 \; \forall k < -1\}.$$

We will prove this. If $s \in N$ we have

$$d(\sigma^{-n_k}(s), \tilde{S}) \leq e^{-\epsilon n_k} \qquad \forall k > 0$$

By the definition of the metric d this implies

$$(\sigma^i(s))_{-2} \neq 1 \quad \text{for} \quad i = n_k, \ldots, n_k + \lceil c\epsilon n_k \rceil - 1 \quad \forall k > 0.$$

where the constant c is independent of ϵ, n_k and $\underline{s}$. Thus we have:

$$N \subseteq \{s | \exists (n_k)_{k \in \mathbb{N}} \longrightarrow \infty \; \forall k > 0 \; : \; (\sigma^i(s))_{-2} \neq 1 \; i = n_k, \ldots, n_k + \lceil c\epsilon n_k \rceil - 1\}.$$

Applying lemma 7.1. of [15] for the ergodic system (Σ, σ, μ) we obtain $\mu(N) = 0$. $\qquad \square$

6 Dimension formulae for ergodic measures

Our results in the laste section demonstrate that we may apply the general dimension theory for hyperbolic systems to the lineare solenoids (Λ_v, f_v, μ_v). By proposition 5.1 and proposition 5.2 our systems fall into the class of generalized hyperbolic attractors in the sense of Schmeling and Troubetzkoy [14]. Usually the dimension theory of ergodic measures is stated in the context of C^2-diffeomorphisms in order to guarantee the existence of Lyapunov exponents and charts. But invertibility and the existence of Lyapunov exponents and charts almost everywhere is enough to apply this theory. We refer to section 4 of [14] for this fact.

We first define here stable partitions $\mathbb{W}^s$ and unstable partitions $\mathbb{W}^u$ of $\mathbb{W}$ by the partition elements

$$\mathbb{W}^s(\mathbf{x}) = \{x\} \times [-1, 1] \times [-1, 1] \qquad \mathbb{W}^u(\mathbf{x}) = \times [-1, 1] \times \{y\} \times \{z\}$$

where $\mathbf{x} = (x, y, z) \in \Lambda_v$. Given $\mu_v \in M(\Lambda_v, f_v)$ we have conditional measures $\mu_v^s(\mathbf{x})$ on the partition $\mathbb{W}^s$ and conditional measures $\mu_v^u(\mathbf{x})$ on the partition $\mathbb{W}^u$. These measures are unique μ_v-almost everywhere fulfilling the relations:

$$\mu_v(B) = \int \mu_v^s(\mathbf{x})(B \cap \mathbb{W}^s(\mathbf{x}))d\mu_v \quad \text{resp.} \quad \mu_v(B) = \int \mu_v^u(\mathbf{x})(B \cap \mathbb{W}^u(\mathbf{x}))d\mu_v$$

for all Borel sets B in $\mathbb{W}$. We refer to [13] for information about conditional measures on measurable partitions.

To formula our next theorem let us denote the entropy of an ergodic measure μ by $h(\mu)$. We recommend [17] for an introduction to theory of this invariant. Moreover we denote the dimension of a measure by $\dim \mu$, so μ is exact dimensional. In the end of the appendix the reader finds an introduction of this quantity.

Applying the dimension theory of hyperbolic systems by Barreira, Schmeling and Pesin [2] and Ledrappier Young [7] to the system (Λ_v, f_v, μ_v) we obtain the following theorem.

Theorem 6.1 *For all $\mu \in M(\Sigma, \sigma)$ the ergodic measures $\mu_v \in M(\Lambda_v, f_v)$ and the condition measures μ_v^s and μ_v^u are exact dimensional with*

$$\dim \mu_v = \dim \mu_v^u + \dim \mu_v^s$$

Moreover we have

$$\dim \mu_v^u = h(\mu)/\log 2$$

and

$$\dim \mu_v^s = \frac{h(\mu)}{-\Xi_{\tau_1,\tau_2}^{\mu}} + (1 - \frac{\Xi_{\beta_1,\beta_2}^{\mu}}{\Xi_{\tau_1,\tau_2}^{\mu}}) \dim pr_y(\mu_v)$$

if $\Xi_{\beta_1,\beta_2}^{\mu} \geq \Xi_{\tau_1,\tau_2}^{\mu}$, resp.

$$\dim \mu_v^s = \frac{h(\mu)}{-\Xi_{\beta_1,\beta_2}^{\mu}} + (1 - \frac{\Xi_{\tau_1,\tau_2}^{\mu}}{\Xi_{\beta_1,\beta_2}^{\mu}}) \dim pr_z(\mu_v)$$

if $\Xi_{\beta_1,\beta_2}^{\mu} < \Xi_{\tau_1,\tau_2}^{\mu}$. Here pr denotes the projection of the measure on second resp. third coordinate axis.

Proof. Exact dimensionality of the measures follows directly from [3] given our results in section five. The dimension formula for μ_v^u follows directly from [7]. For the second formula we need an additional argument. If $\Xi_{\beta_1,\beta_2}^{\mu} \geq \Xi_{\tau_1,\tau_2}^{\mu}$

$$\mathbb{W}^{ss}(\mathbf{x}) = \{x\} \times \{y\} \times [-1, 1]$$

forms the strong stable partition. We have conditional measures $\mu_v^{ss}(\mathbf{x})$ on $\mathbb{W}^{ss}$. These measures are unique μ_v-almost everywhere fulfilling the relation:

$$\mu_v(B) = \int \mu_v^{ss}(\mathbf{x})(B \cap \mathbb{W}^{ss}(\mathbf{x}))d\mu_v$$

for all Borel sets B in $\mathbb{W}$. From the uniqueness of the conditional measures we have for μ_v-almost all $\mathbf{x}$

$$\mu_v^s(\mathbf{x})(B) = \int \mu_v^{ss}(\mathbf{x})(B \cap \mathbb{W}^{ss}(\mathbf{x}) dpr_y(\mu_v)$$

for all Borel sets B in $W^s(\mathbf{x})$. This statement means that the transversal measures in the sense of [7] of the nested partitions $\mathbb{W}^s$ and $\mathbb{W}^{ss}$ are given by $pr_y\mu_v$. Now the second formula follows from [7]. The third formula is proved the same way just noticing that the strong stable partition is given by

$$\mathbb{W}^{ss}(\mathbf{x}) = \{x\} \times [-1, 1] \times \{z\}$$

in this case. □

The formula for the condition measures μ_v^s in theorem 6.1 is known in the dimension theory as *Ledrappier-Young formula*. For Bernoulli measures $b^p \in M(\Sigma, \sigma)$ we get by theorem 6.1 the following explicit dimension formulas for the measures $b_v^p \in M(\Lambda_v, f_v)$.

Corollary 6.1

$$\dim b_v^p = \frac{p \log p + (1-p) \log(1-p)}{\log 2} + \frac{p \log p + (1-p) \log(1-p)}{p \log \tau_1 + (1-p) \log \tau_2}$$

$$+(1 - \frac{p \log \beta_1 + (1-p) \log \beta_2}{p \log \tau_1 + (1-p) \log \tau_2}) \dim \pi_{\beta_1, \beta_2}(b^p).$$

if $p \log \beta_1 + (1-p) \log \beta_2 \geq p \log \tau_1 + (1-p) \log \tau_2$

$$\dim b_v^p = \frac{p \log p + (1-p) \log(1-p)}{\log 2} + \frac{p \log p + (1-p) \log(1-p)}{p \log \beta_1 + (1-p) \log \beta_2}$$

$$+(1 - \frac{p \log \tau_1 + (1-p) \log \tau_2}{p \log \beta_1 + (1-p) \log \beta_2}) \dim \pi_{\tau, \tau_2}(b^p).$$

if $p \log \beta_1 + (1-p) \log \beta_2 < p \log \tau_1 + (1-p) \log \tau_2$.

Proof. It is well known in ergodic theory that

$$h(b^p) = -(p \log p + (1-p) \log(1-p))$$

see [3]. Furthermore we obviously have

$$\Xi_{\gamma_1, \gamma_2}^{b^p} = (p \log \gamma_1 + (1-p) \log \gamma_2)$$

Thus it remains to show $pr_y(b_v^p) = \pi_{\beta_1, \beta_2}(b^p)$ (resp. $pr_z(b_v^p) = \pi_{\tau_1, \tau_2}(b^p)$) but this is immediate from the product property of Bernoulli measures and the definition of the coding map π_v in section four. □

The self similar Bernoulli measures $\pi_{\beta_1, \beta_2}(b^p)$ resp. $\pi_{\beta_1, \beta_2}(b^p)$ where extensively studied in [9]. We have results on absolut continuity, singularity and the dimension of this measures.

7 Measures of full Dimension

In this section we will ask the question whether there exists ergodic measures of full dimension for linear solenoids. Our result is that in general such a measure does not exist, the dimension of a attractor can not even be approximated by the dimension of ergodic measure. On the other hand we will proof that under the assumption of certain symmetries of the system the equal weighted Bernoulli measure on the attractor has full dimension. Our theorem is a consequence of both our results on the dimension of the attractor in section three and our results on the dimension of ergodic measures in section six.

Theorem 7.1 *For all $v = (\beta_1, \beta_2, \tau_1, \tau_2) \in (0,1)$ with $\tau_1 + \tau_2 < \beta_1 + \beta_2 < 1$ we have*

$$\dim b_v^{0.5} = \dim_B \Lambda_v = \dim_H \Lambda_v$$

if $\beta_1 = \beta_2$ and

$$\sup\{\dim \mu | \mu \in M(\Lambda_v, f_v)\} < \dim_B \Lambda_v = \dim_H \Lambda_v$$

if $\beta_1 \neq \beta_2$.

For almost all $\beta_1, \beta_2 \in (0, 0.649)$ with $\beta_1 + \beta_2 \geq 1$ and all $\tau_1, \tau_2 \in (0,1)$ with $\tau_1 + \tau_2 < \beta_1 + \beta_2$ we have

$$\dim b_v^{0.5} = \dim_B \Lambda_v = \dim_H \Lambda_v$$

if $\log_{\tau_2} 2\beta_2 = \log_{\tau_1} 2\beta_1$ and

$$\sup\{\dim \mu | \mu \in M(\Lambda_v, f_v)\} < \dim_B \Lambda_v = \dim_H \Lambda_v$$

if $\log_{\tau_2} 2\beta_2 \neq \log_{\tau_1} 2\beta_1$.

Proof. First assume $\tau_1 + \tau_2 < \beta_1 + \beta_2 < 1$ and let d be the solution of $\beta_1^d + \beta_2^d = 1$. In the case $\beta_1 = \beta_2 = 1/2$ we have $\beta_1^d = \beta_2^2 = 1/2$. It is well know in dimensiontheory that $\dim \pi_{\beta_1, \beta_2}(b^{0.5}) = d$, see [12]. Thus we get by Corollary 6.1

$$\dim b_v^{0.5} = 1 + \frac{\beta_1^d \log \beta_1^d + \beta_2^d \log \beta_2^d}{\beta_1^d \log \tau_1 + \beta_2^d \log \tau_2} + (1 - \frac{\beta_1^d \log \beta_1 + \beta_2^d \log \beta_2}{\beta_1^d \log \tau_1 + \beta_2^d \log \tau_2})d = d + 1$$

But this is by theorem 3.1 the dimension of the attractor Λ_v. In the case $\beta_1 \neq \beta_2$ we know that $\dim \pi_{\beta_1, \beta_2}(b^{0.5}) < d$, see [12]. Consider a sequence $\mu_n \in M(\Sigma, \sigma)$ with $\mu_n \longmapsto b^{0.5}$. By theorem 6.1 we get

$$\overline{\lim}_{n \longrightarrow \infty} \dim_H (\mu_n)_v^s < d + \frac{-2 \log 2 - \log \beta_1^d - \log \beta_2^d}{\log \tau_1 + \log \tau_2} < d$$

We thus see that there is a weak* neighborhood U of $b^{0.5}$ with

$$\sup\{\dim \mu_v^s | \mu \in U\} < d$$

On the other hand it is well known in ergodic theory that

$$\sup\{h(\mu) | \mu \in M(\Sigma, \sigma) \backslash U\} < \log 2$$

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and together we get by theorem 3.1

$$\sup\{\dim \mu_v | \mu \in M(\Sigma, \sigma)\} < d + 1$$

Our claim now follows from theorem 3.1 and proposition 4.2.

Now assume $\tau_1 + \tau_2 < \beta_1 + \beta_2 \geq 1$ and let d be the solution of $\beta_1 \tau^d + \beta_2 \tau_2^d = 1$. In the case $\log_{\tau_2} 2\beta_2 = \log_{\tau_1} 2\beta_1$ we have $\beta_1 \tau_1^d = \beta_2 \tau_2^d = 1/2$. From [9] we know that for almost all $\beta_1, \beta_2 \in (0, 0.649)$ we have $\dim \pi_{\beta_1, \beta_2}(b^{0.5}) = 1$. Thus we get by Corollary 6.1

$$\dim_H b_v^{0.5} = 2 + \frac{\beta_1 \tau_1^d \log \beta_1 \tau_1^d + \beta_2 \tau_2^d \log \beta_2 \tau_2^d - (\beta_1 \tau_1^d \log \beta_1 + \beta_2 \tau_2^d \log \beta_2)}{\beta_1 \tau_1^d \log \tau_1 + \beta_2 \tau_2^d \log \tau_2}$$

$$= 2 + \frac{\beta_1 \tau_1^d \log \tau_1^d + \beta_2 \tau_2^d \log \tau_2^d}{\beta_1 \tau_1^d \log \tau_1 + \beta_2 \tau_2^d \log \tau_2} = d + 2.$$

But this is by theorem 3.1 the dimension of the attractor Λ_v. Now let the $\log_{\tau_2} 2\beta_2 \neq \log_{\tau_1} 2\beta_1$. We get by theorem 6.1 the following upper estimate

$$\dim_H \mu_v^s \leq 1 - \frac{h_{(\mu)} + \Xi_{\beta_1, \beta_2}^\mu}{\Xi_{\tau_1, \tau_2}^\mu}$$

for all $\mu \in M(\Sigma, \sigma)$. If $\mu_n \longmapsto b^{0.5}$ this yield

$$\overline{\lim}_{n \longrightarrow \infty} \dim_H (\mu_n)_v^s \leq 1 - \frac{\log 2 + 0.5 \log \beta_1 + 0.5 \log \beta_2}{0.5 \log \tau_1 + 0.5 \log \tau_2} < d + 1.$$

We thus again see that there is a weak* neighborhood U of $b^{0.5}$ with

$$\sup\{\dim \mu_v^s | \mu \in U\} < d + 1$$

Now our claim follows in the same way as in the case $\beta_1 + \beta_2 < 1$. □

Clearly the statement of theorem 7.1 holds as well if we interchange the role of β and τ. At the end of this work we again remark that it should be possible to replace the bound 0.649 in theorem 7.1 by 1 using new ideas on continuity of self-similar Bernoulli measures $\pi_{\beta_1, \beta_2}(b^{0.5})$ in the case $\beta_1 + beta_2 \geq 1$.

8   Appendix: General facts in dimension theory

We will here first define the most important quantities in dimension theory and collect some basic facts we need. We refer to the book of Falconer [4] and the book of Pesin [12] for a more detailed discussion of dimension theory.

Let $Z \subseteq \mathbb{R}^q$. We define the *s-dimensional Hausdorff measure* $H^s(Z)$ of Z by

$$H^s(Z) = \lim_{\lambda \longrightarrow 0} \inf\{\sum_{i \in I} (\mathrm{diam} U_i)^s | Z \subseteq \bigcup_{i \in I} U_i \text{ and } \mathrm{diam}(U_i) \leq \lambda\}.$$

The *Hausdorff dimension* $\dim_H Z$ of Z is given by

$$\dim_H Z = \sup\{s|H^s(Z) = \infty\} = \inf\{s|H^s(Z) = 0\}.$$

Let $N_\epsilon(Z)$ be the minimal number of balls of radius ϵ that are needed to cover Z. We define the *upper box-counting dimension* $\overline{\dim}_B$ resp. *lower box-counting dimension* $\overline{\dim}_B$ of Z by

$$\overline{\dim}_B Z = \overline{\lim}_{\epsilon \longrightarrow 0} \frac{\log N_\epsilon(Z)}{-\log \epsilon} \qquad \underline{\dim}_B Z = \underline{\lim}_{\epsilon \longrightarrow 0} \frac{\log N_\epsilon(Z)}{-\log \epsilon}.$$

Proposition 8.1 *If $Z \subseteq \mathbb{R}^q$ and $I \subseteq \mathbb{R}$ is an interval then $\dim_H(Z \times I) = \dim_H +1$ and $\dim_B(Z \times I) = \dim_B +1$ holds for both upper an lower dimension.*

The statement for the Hausdorff dimension follows from proposition 7.4. of [4] and the statement for the box-counting dimension is easy to see using 3.1. of [4].

Now let μ be a Borel probability measure on $\mathbb{R}^q$. We define the dimensional theoretical quantities for μ by

$$\dim_H \mu = \inf\{\dim_H Z|\mu(Z) = 1\}$$

and

$$\overline{\dim}_B \mu = \lim_{\rho \longrightarrow 0} \inf\{\overline{\dim}_B Z|\mu(Z) \geq 1 - \rho\}.$$

We introduce one more notion of dimension for a measure μ. The *upper local dimension* $\overline{d}(x, \mu)$ resp. *lower local dimension* $\underline{d}(x, \mu)$ of the measure μ in a point x is defined by

$$\overline{d}(x, \mu) = \overline{\lim}_{\epsilon \longrightarrow 0} \frac{\mu(B_\epsilon(x))}{\log \epsilon} \qquad \underline{d}(x, \mu) = \underline{\lim}_{\epsilon \longrightarrow 0} \frac{\mu(B_\epsilon(x))}{\log \epsilon}.$$

The relations between the local dimension and the other notion of dimension of measures are described in the following theorem:

Theorem 8.1 *Let μ be a Borel probability measure on $\mathbb{R}^q$. If*

$$\overline{d}(x, \mu) = \underline{d}(x, \mu) = c$$

almost for holds for almost all $x \in \mathbb{R}^q$ we have

$$dim_H \mu = \dim_B \mu = c.$$

A proof of this theorem is contained in the work of Young [18]. If the condition in this theorem holds, the measure μ is called *exact dimensional* and the common value of the dimensions is denoted by $\dim \mu$ and maybe called the fractal dimension of the measure.

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