# Exceptions in the domain of generic absolute continuity of non-homogeneous self-similar measures

Jörg Neunhäuserer
Technical University of Braunschweig
joerg.neunhaeuserer@web.de

**Abstract**

Non-homogeneous self-similar measures are generically absolute continuous in the domain of parameters for which the similarity dimension is larger than one, see [10]. Using certain algebraic curves we construct here exceptional singular non-homogeneous self-similar measures in this domain.

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## 1 Introduction and main result

For $\beta_1, \beta_2 \in (0,1)$ we consider the similarities on $\mathbb{R}$ given by

$$T_1(x) = \beta_1 x + \beta_1 \text{ and } T_2(x) = \beta_2 x - \beta_2.$$

It is well known that for probability $p \in (0,1)$ there is an unique self-similar Borel probability measure $\mu^p_{\beta_1,\beta_2}$ on $\mathbb{R}$ with

$$\mu^p_{\beta_1,\beta_2} = pT_1(\mu^p_{\beta_1,\beta_2}) + (1-p)T_{-1}(\mu^p_{\beta_1,\beta_2}),$$

see [5]. If $\beta_1 = \beta_2 = \beta \in (0,1)$ the measure $\mu^p_\beta = \mu^p_{\beta,\beta}$ is an infinit convolved Bernoulli measure. These measure were intensively studied since the pioneering work of Erdös [2, 3] with milestone results of Solomyak [12], Hochman [4], Shmerkin [11] and Varju [13]. In the case $\beta_1 \neq \beta_2$ the measures $\mu^p_{\beta_1,\beta_2}$ are now usually called non-homogeneous self-similar.[1] The similarity dimension of the function system $\{T_1, T_2\}$, with respect to the probability vector $(p, (1-p))$, is defined by

$$SD^p_{\beta_1,\beta_2} = \frac{-p\log(p) - (1-p)\log(1-p)}{-p\log\beta_1 - (1-p)\log\beta_2}.$$

The measures $\mu^p_{\beta_1,\beta_2}$ are of pure type, they are either totally singular or absolutely continuous and even equivalent to the Lebesgue measure in the second case, see proposition

[1] We were used to call these measure non-uniform self-similar but adopt the terminology from the decisive book [1].

3.1 in [9]. Since the work of Hutchinson [5] we know that Hausdorff dimension of $\mu^p_{\beta_1,\beta_2}$ is bounded by the similarity dimension

$$\dim_H \mu^p_{\beta_1,\beta_2} \leq SD^p_{\beta_1,\beta_2}.$$

Hence the measures are singular if $SD^p_{\beta_1,\beta_2} < 1$. The main result on absolut continuity reads as follows:

**Theorem 1.1** *For all $p \in (0,1)$ and almost all $\beta_1, \beta_2 \in (0,1)$ with $SD^p_{\beta_1,\beta_2} > 1$ the measure $\mu^p_{\beta_1,\beta_2}$ is absolutely continuous.*

We proved this result for $\beta_1, \beta_2 \in (0, 0.649)$ in [6] using transversality techniques which lead to the crude bound 0.649. Ngai and Wang [8] proved a similar result using transversality. Saglietti, Shmerkin and Solomyak [10] succeeded in removing the bound 0.649 by an improvement of the classical tranversality method. For infinit convolved Bernoulli measure $\mu^p_\beta$ we have an even stronger result: The set of expectations to absolute continuity in the domain $\beta > p^p(1-p)^(1-p)$ has Hausdorff measure zero, see [11]. The only known exceptions found by Erdös [2] are algebraic numbers $\beta \in (0.5, 1)$, which are reciprocals of Pisot numbers.[2] In this case we have $\dim_H \mu^p_\beta < 1$. In the non-homogeneous case we proved in [7] the existence of $\beta_1 \neq \beta_2$ with $\beta_1\beta_2 > 1/4$ such that $\dim_H \mu^{0.5}_{\beta_1,\beta_2} < 1$, without construction explicit examples. Here we construct explicit algebraic curves with points $\beta_1, \beta_2 \in (0,1)$ such that $SD^p_{\beta_1,\beta_2} > 1$ but $\dim_H \mu^p_{\beta_1,\beta_2} < 1$ for some $p \in (0,1)$. These are exceptions in the domain of generic absolute continuity of non-homogeneous self-similar measures.

For $n \geq 3$ consider a finite sequence $s \in \{-1,1\}^n$. For $1 \leq k \leq n$ let $\sharp_k(s)$ be the number of entries in $(s_1, s_2, \ldots, s_k)$ that are 1 and $\tilde{\sharp}_k(s) = k - \sharp_k(s)$ the number of entries that are $-1$. For two sequences $s, t \in \{-1,1\}^n$ with $s \neq t$ and $\sharp_n(s) = \sharp_n(t)$ we define an algebraic curve $c_{s,t}$ by the algebraic equation

$$\sum_{k=1}^{\infty} s_k x^{\sharp_k(s)} y^{\tilde{\sharp}_k(s)} - t_k x^{\sharp_k(t)} y^{\tilde{\sharp}_k(t)} = 0$$

in $\mathbb{R}^2$. Let

$$\mathfrak{C} = \{c_{s,t} | s, t \in \{-1,1\}^n, s \neq t, \sharp_n(s) = \sharp_n(t), n \geq 3\}$$

be the set of all of such curves. With this notations we formulate our main result.

**Theorem 1.2** *If $(\beta_1, \beta_2) \in (0,1)^2$ with $\beta_1 + \beta_2 > 1$ is a point on a curve in $\mathfrak{C}$, then there is a $p \in (0,1)$ such that $SD^p_{\beta_1,\beta_2} > 1$, but the measure $\mu^p_{\beta_1,\beta_2}$ is singular with $\dim_H \mu^p_{\beta_1,\beta_2} < 1$.*

[2] A Pisot numbers is an algebraic integer $\alpha > 1$ with all its algebraic conjugates inside the unit circle.

We prove this theorem in the next section. So far it is open if their are any curves in $\mathfrak{C}$ that have points in the open rectangle

$$R = \{(\beta_1, \beta_2) \in (0,1)^2 | \beta_1 + \beta_2 > 1\}.$$

For $n = 3$ and $n = 4$ some calculation show that there do not exists such curves. For $n = 5$ we consider $s = (1, -1, -1, -1, 1)$ and $t = (-1, 1, 1, -1, -1)$, which lead to the algebraic equation

$$2x^2y^3 + x^2y^2 - x^2y - xy^3 - xy^2 - 2xy + x + y = 0.$$

The corresponding algebraic curve $c_{s,t,}$ has approbate points, see figure 1. For $n = 5$ this is the only curve with the required properties found. In the last section of the paper we will give suffizient conditions on the sequences $s, t \in \{-1, 1\}^n$ such that $c_{s,t} \cap R \neq \emptyset$. These conditions especially apply to the example described here. Examples of curves which have the required properties with $n = 6$ will be given there.

## 2 Proof of theorem 1.2

We first estimate the Hausdorff dimensions of the measures $\mu^p_{\beta_1,\beta_2}$ under the assumptions of theorem 1.2.

**Proposition 2.1** *If $(\beta_1, \beta_2) \in (0,1)^2$ is a point on a curve in $\mathfrak{C}$ than*

$$\dim_H \mu^p_{\beta_1,\beta_2} \leq \widehat{SD}^p_{\beta_1,\beta_2} < SD^p_{\beta_1,\beta_2}$$

*for all $p \in (0,1)$ and $\widehat{SD}^p_{\beta_1,\beta_2}$ is continuous in $p$.*

**Proof.** For $r \in \{-1, 1\}^n$ we set

$$T_r(x) = T_{r_n} \circ T_{r_2} \circ \cdots \circ T_{r_1}(x)$$

$$= \beta_1^{\sharp_n(r)} \beta_2^{\tilde{\sharp}_n(r)} x + \sum_{k=1}^{\infty} r_k \beta_1^{\sharp_k(r)} \beta_2^{\tilde{\sharp}_k(r)}.$$

Note that if $(\beta_1, \beta_2) \in (0,1)^2$ is a point on a curve $c_{s,t} \in \mathfrak{C}$ for $s, t \in \{-1, 1\}^n$, than $T_s = T_t$. By [5] there is a unique Borel probability measure $\mu$ on $\mathbb{R}$ fulfilling

$$\mu = \sum_{r \in \{-1,1\}^n} p^{\sharp_n(r)} (1-p)^{\tilde{\sharp}_n(r)} T_r(\mu)$$

$$= \sum_{r \in \{-1,1\}^n \backslash \{s,t\}} p^{\sharp_n(r)} (1-p)^{\tilde{\sharp}_n(r)} T_r(\mu) + (p^{\sharp_n(s)} (1-p)^{\tilde{\sharp}_n(s)} + p^{\sharp_n(t)} (1-p)^{\tilde{\sharp}_n(t)}) T_s(\mu)$$

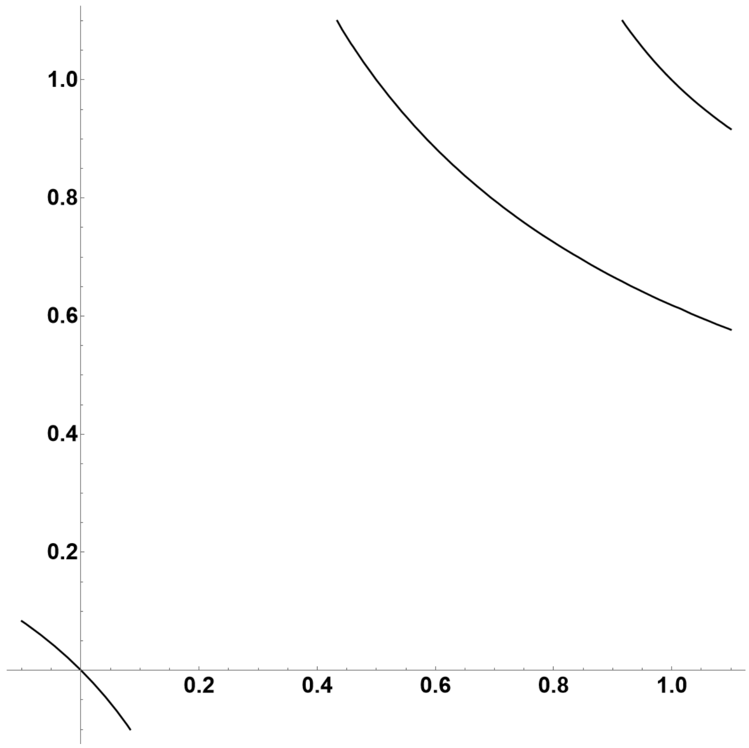


Figure 1: The curve $c_{s,t}$ for s=(1,-1,-1,-1,1) and $t = (-1, 1, 1, -1, -1)$

and we obviously have $\mu = \mu^p_{\beta_1,\beta_2}$. The similarity dimension of the function system $\{T_r | r \in \{-1,1\}^n \backslash \{s,t\}\} \cup \{T_s\}$ with respect to the probability vector that is summing up the probability of $s$ and $t$,

$$p_{s,t} = (p^{\sharp_n(s)}(1-p)^{\tilde{\sharp}_n(s)} + p^{\sharp_n(t)}(1-p)^{\tilde{\sharp}_n(t)})$$

is given by

$$\widehat{SD}^p_{\beta_1,\beta_2} = \frac{-\sum_{r\in\{-1,1\}^n\backslash\{s,t\}} p^{\sharp_n(r)}(1-p)^{\tilde{\sharp}_n(r)} \log(p^{\sharp_n(r)}(1-p)^{\tilde{\sharp}_n(r)}) - p_{s,t}\log(p_{s,t})}{-\sum_{r\in\{-1,1\}^n\backslash\{s,t\}} p^{\sharp_n(r)}(1-p)^{\tilde{\sharp}_n(r)} \log(\beta_1^{\sharp_n(r)}\beta_2^{\tilde{\sharp}_n(r)}) - p_{s,t}\log(\beta_1^{\sharp_n(s)}\beta_2^{\tilde{\sharp}_n(s)})}$$

$$= \frac{-\sum_{r\in\{-1,1\}^n\backslash\{s,t\}} p^{\sharp_n(r)}(1-p)^{\tilde{\sharp}_n(r)} \log(p^{\sharp_n(r)}(1-p)^{\tilde{\sharp}_n(r)}) - p_{s,t}\log(p_{s,t})}{-p\log\beta_1 - (1-p)\log\beta_2}$$

$$< \frac{-p\log(p) - (1-p)\log(1-p)}{-p\log\beta_1 - (1-p)\log\beta_2} = SD^p_{\beta_1,\beta_2}.$$

The inequality here is due to

$$-p_{s,t}\log(p_{s,t}) < -(p^{\sharp_n(s)}(1-p)^{\tilde{\sharp}_n(s)}\log((p^{\sharp_n(s)}(1-p)^{\tilde{\sharp}_n(s)}) - p^{\sharp_n(t)}(1-p)^{\tilde{\sharp}_n(t)}\log(p^{\sharp_n(t)}(1-p)^{\tilde{\sharp}_n(t)}).$$

The proposition now follows from [5]. □

To complete the proof of theorem 1.2 it remains to show:

**Proposition 2.2** *If $(\beta_1, \beta_2) \in (0,1)^2$ with $\beta_1 + \beta_2 > 1$ is a point on a curve in $\mathfrak{C}$ than there is an $p \in (0,1)$ such that $\widehat{SD}^p_{\beta_1,\beta_2} < 1$ and $SD^p_{\beta_1,\beta_2} > 1$. Here $\widehat{SD}^p_{\beta_1,\beta_2} < 1$ is given by the last proposition.*

**Proof.** Since $\beta_1 + \beta_2 > 1$ there is an $d > 1$ such that $\beta_1^d + \beta_2^d = 1$. Let $p_M = \beta_1^d \in (0,1)$. A simple calculation shows

$$SD^{p_M}_{\beta_1,\beta_2} = d > 1.$$

Since

$$\lim_{p \to 0} SD^p_{\beta_1,\beta_2} = \lim_{p \to 1} SD^p_{\beta_1,\beta_2} = 0$$

there is a $p_1 \in (0,1)$ such that $SD^{p_1}_{\beta_1,\beta_2} = 1$. Since $\widehat{SD}^p_{\beta_1,\beta_2} < SD^p_{\beta_1,\beta_2}$ for all $p \in (0,1)$ by proposition 2.1 and $SD^p_{\beta_1,\beta_2}$ and $\widehat{SD}^p_{\beta_1,\beta_2}$ are continuous in $p$ there is an $p$ near $p_1$ such that $\widehat{SD}^p_{\beta_1,\beta_2} < 1$ and $SD^p_{\beta_1,\beta_2} > 1$. □

# 3 Algebraic curves given the exceptions

To apply theorem 2.1 we need sequences $s, t \in \{-1,1\}^n$ such that the algebraic curves $c_{s,t} \in \mathfrak{C}$, defined in the first section, have points in the open rectangle

$$R = \{(x,y) \in (0,1)^2 | x + y > 1\}.$$

We find a sufficient condition on the sequences that guarantees the existences of such points.

**Proposition 3.1** *For $n \geq 3$ let $s, t \in \{-1,1\}^n$ with $s \neq t$ and $\sharp_n(s) = \sharp_n(t)$. Assume that $s$ begins with $(1,-1)$ and $t$ begins with $(-1,1)$. If*

$$\sum_{k=1}^{\infty} s_k \sharp_k(s) > \sum_{k=1}^{\infty} t_k \sharp_k(t),$$

*we have $c_{s,t} \cap R \neq \emptyset$.*

**Proof.** Let

$$f(x,y) = \sum_{k=1}^{\infty} s_k x^{\sharp_k(s)} y^{\tilde{\sharp}_k(s)} - t_k x^{\sharp_k(t)} y^{\tilde{\sharp}_k(t)}$$

and

$$f(x) = f(x,1) = \sum_{k=1}^{\infty} s_k x^{\sharp_k(s)} - t_k x^{\sharp_k(t)}.$$

By the first assumption we have $f(0) = 1$ and since $\sharp_n(s) = \sharp_n(t)$ we have $f(1) = 0$. By the second assumption we obtain

$$f'(1) = \sum_{k=1}^{\infty} s_k \sharp_k(s) - \sum_{k=1}^{\infty} t_k \sharp_k(t) > 0,$$

which implies $f(x) < 0$ for some $x \in (0,1)$. It follows that there is an $x_0 \in (0,1)$ and $\delta > 0$ such that $f(x_0,1) = f(x_0) = 0$ and $f(x) > 0$ for $x \in (x_0-\delta, x_0)$ and $f(x) < 0$ for $x \in (x_0, x_0+\delta)$. Hence by continuity for all suffizient small $\epsilon > 0$ there exists a $x_- \in (x_0-\delta, x_0)$ with $f(x_-, 1-\epsilon) < 0$ and $x_+ \in (x_0, x_0+\delta)$ with $f(x_+, 1-\epsilon) < 0$. Again we continuity there is an $x_\epsilon \in (x_-, x_+)$ with $f(x_\epsilon, 1-\epsilon) =$. Moreover if $\epsilon$ is suffizient small $(x_\epsilon, 1-\epsilon) \in R$. □

This proposition applies to the sequences $s,t$ with $n = 5$ given in the first section of the paper. For $n = 6$ we have the following examples:

$$s = (1,-1,-1,1,1,1) \quad t = (-1,1,1,1,1,-1),$$

$$s = (1,-1,1,-1,1,1) \quad t = (-1,1,1,1,1,-1),$$

$$s = (1,-1,-1,-1,1,1) \quad t = (-1,1,-1,1,1,-1),$$

$$s = (1,-1-,1,-1,1,1) \quad t = (-1,1,1,-1,1,-1),$$

given the algebraic equations

$$2x^4y^2 - x^4y + x^3y^2 - x^3y + x^2y^2 - x^2y - xy^2 - 2xy + x + y = 0,$$

$$2x^4y^2 - x^4y + x^3y^2 - x^3y - x^2y^2 - 2xy + x + y = 0,$$

$$2x^3y^3 - x^3y^2 + x^2y^3 - x^2y^2 - xy^3 - 2xy + x + y = 0,$$

$$2x^3y^3 - x^3y^2 + x^2y^3 + x^2y^2 - x^2y - xy^3 - xy^2 - 2xy + x + y = 0.$$

We do not know if the condition given in the last proposition is (up to symmetries) necessary to guarantee the existentes of appropriate curves. We leave this question to the reader.

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