A new family of expansions of real numbers

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Abstract

For $\alpha > 1$ we represent a real number in $(0, 1]$ in the form

$$\sum_{i=1}^{\infty}(\alpha-1)^{i-1}\alpha^{-(d_1+\cdots+d_i)}$$

with $d_i \in \mathbb{N}$. We discuss ergodic theoretical and dimension theoretical aspects of this expansion. Furthermore we study their base-change-transformation.

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1 Introduction

Expansion of real numbers, like classical b-adic expansions and continued fraction expansions, are a central topic in the metric theory of numbers. We like to remind the reader of some classical results. Borel [6] proved that almost all real numbers are normal to all bases b, in the b-adic expansions all digits appear with same frequency. It was Khinchin [17] who proved, that for almost all real numbers the asymptotic geometric mean of the digits of the continued fraction expansion is a universal constant, now called Khinchin's constant. Such result are today a part of ergodic theory. Hausdorff [19] introduced his dimension to distinguish the size of real numbers with digits in the b-adic expansion from a proper subset of all digits. Furthermore Besicovitch [5] and Eggleston [8] found the Hausdorff dimension of the set of real numbers with given frequencies of digits in the b-adic expansion. The dimension theory of continued fraction expansions goes back to Jarnick [11], who shows that the set of continued fractions with bounded digits has Hausdorff dimension one, although this set has Lebesgue measure zero.

Beside b-adic expansions and continued fraction expansions many other expansions of real number are studied. We have ergodic and dimension theoretical results on β-expansions [29, 2], Cantor series expansions [9, 16, 1], Engel-expansions [10, 31, 12], Lüroth-expansions [20, 4], multiple-base expansions [25, 25, 22, 18] and others.

In this paper we introduce a new family of expansions of real numbers which is given essentially by the expression in the abstract. We call this expansion α-expansion and describe this expansion in detail in the next section. In section 4 we consider ergodic aspects of α-expansions and proof an analogon of Borels and of Khinchins theorem. In the next section we study dimension theoretical aspects of α-expansions. We proof results similar to the theorems of Hausdorff and Jarnik and similar to the theorems of Besicovitch and Eggleston. In the last section we discuss the base-change-transformation for α-expansions. It turns out that the transformation is strictly monoton and continuous with an infinit self-similar graph of Hausdorff dimension one. Furthermore generically the upper derivative of this transformation is infinite and the lower derivative is zero.

2 The expansions

Let $\alpha > 1$ be a real number. For an infinite sequence $(d_i) \in \mathbb{N}^{\mathbb{N}}$ we set

$$\langle d_i \rangle_\alpha = \langle d_1, d_2, \dots \rangle_\alpha = \sum_{i=1}^{\infty} (\alpha - 1)^{i-1} \alpha^{-(d_1+\cdots+d_i)}.$$

Let $\pi : \mathbb{N}^{\mathbb{N}} \to (0,1]$ by given by $\pi((d_i)) = \langle d_i \rangle_\alpha$. Since

$$\sum_{i=1}^{\infty} (\alpha - 1)^{i-1} \alpha^{-(d_1+\cdots+d_i)} \leq \sum_{i=1}^{\infty} (\alpha - 1)^{i-1} \alpha^{-i} = 1$$

the map is well defined. We will use an other discription of the map π. For $i \in \mathbb{N}$ let $T_i : [0,1] \to [0,1]$ be given by

$$T_i(x) = \frac{\alpha - 1}{\alpha^i} x + \frac{1}{\alpha^i}.$$

We have:

Lemma 2.1

$$\pi((d_i)) = \lim_{n \to \infty} T_{d_n} \circ \cdots \circ T_{d_1}(1).$$

Proof. By induction we get

$$T_{d_n} \circ \cdots \circ T_{d_1}(x) = \frac{(\alpha - 1)^n}{\alpha^{d_1+\cdots+d_n}} x + \sum_{i=1}^{n} (\alpha - 1)^{i-1} \alpha^{-(d_1+\cdots+d_i)}.$$

The result follows since the maps T_i are contractions for all $i \in \mathbb{N}$. □

Using this lemma we easily obtain:

Proposition 2.1 *π is a bijection.*

Proof. We have

$$\bigcup_{i=1}^{\infty} T_i((0,1]) = \bigcup_{i=1}^{\infty} \left(\frac{1}{\alpha^i}, \frac{1}{\alpha^{i-1}} \right] = (0,1]$$

hence π is surjective. Moreover

$$T^i((0,1]) \cap T^j((0,1]) = \left(\frac{1}{\alpha^i}, \frac{1}{\alpha^{i-1}}\right] \cap \left(\frac{1}{\alpha^j}, \frac{1}{\alpha^{j-1}}\right] = \emptyset$$

it $i \neq j$, hence π is injective. □

By this proposition we have an unique expansion of the form $x = \langle d_i \rangle_\alpha$ with $(d_i) \in \mathbb{N}^\mathbb{N}$ for every $x \in (0,1]$. Thus we may define:

Definition 2.1 For $\alpha > 1$ the sequence $(d_i) \in \mathbb{N}^\mathbb{N}$ with $x = \langle d_i \rangle_\alpha$ is the α-expansion of x.

3 Ergodic theoretical aspects

We use here a dynamical approach to proof results on the distribution of digits of α-expansions. The reader who is not familiar with basic ergodic theory should consider our essential [27] or [7].
Let $\sigma : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ given by $\sigma(d_k) = d_{k+1}$ be the shift map. We define the corresponding map on $T : (0,1] \to (0,1]$, with respect to the coding map π, by

$$T(x) = T_i^{-1}(x) \text{ for } x \in T_i((0,1]) = \left(\frac{1}{\alpha^i}, \frac{1}{\alpha^{i-1}}\right]$$

for $i \in \mathbb{N}$. By lemma 2.1 we have the conjugation

$$T \circ \pi = \pi \circ \sigma$$

and obtain

Lemma 3.1 *For $x = \langle d_i \rangle_\alpha \in (0,1]$ we obviously have $d_j = d \in \mathbb{N}$, if and only if*

$$T^{j-1}(x) \in T_d((0,1]) = \left(\frac{1}{\alpha^d}, \frac{1}{\alpha^{d-1}}\right].$$

Proof. Obviously $x = \langle d_i \rangle_\alpha \in T_d((0,1])$, if and only if $d_1 = d$. Hence

$$T^{j-1}(x) = T^{j-1}(\langle d_i \rangle_\alpha) = \langle \sigma^{j-1}(d_i) \rangle_\alpha \in T_d((0,1]),$$

if and only if $d_j = d$. □

To order to apply Birkhoff's ergodic theorem we state:

Proposition 3.1 *The Lebesgue measure $\mathfrak{L}$ is ergodic with respect to T.*

Proof. For an open interval $(a,b) \subseteq [0,1]$ we have

$$\mathfrak{L}(T^{-1}((a,b))) = \mathfrak{L}\left(\bigcup_{i=1}^{\infty}(\frac{\alpha-1}{\alpha^i}a + \frac{1}{\alpha^i}, \frac{\alpha-1}{\alpha^i}b + \frac{1}{\alpha^i}\right)$$

$$= \sum_{i=1}^{\infty}\frac{\alpha-1}{\alpha^i}\mathfrak{L}\left((a,b))\right) = b-a = \mathfrak{L}((a,b)).$$

Hence $\mathfrak{L}(T^{-1}(B)) = \mathfrak{L}(B)$ for all Borel sets $B \subseteq (0,1]$, which means that $\mathfrak{L}$ is invariant under T. We could now directly proof that $\mathfrak{L}$ is ergodic with respect to T using Lebesgue density theorem, but this is not necessary. It is well known that for piecewise smooth expanding interval maps the absolutely continuous invariant measure is in fact ergodic, see [21]. □

Now we consider the frequency of a digit $d \in \mathbb{N}$ in the α-expansion of $x = \langle d_i \rangle_\alpha \in (0,1]$,

$$\mathfrak{f}_{d,\alpha}(x) := \lim_{n\to\infty}\frac{\sharp\{j \in \{1,\ldots,n\}|d_j = d\}}{n}.$$

Generically this frequencies are geometrically distributed:

Theorem 3.1 *Let $\alpha > 1$. For almost all $x \in (0,1]$ we have $\mathfrak{f}_{d,\alpha}(x) = (\alpha-1)/\alpha^d$ for all $d \in \mathbb{N}$.*

Proof. Let χ_A be the characteristic function of $A \subseteq \mathbb{R}$, that is $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ else. For $x = \langle d_i \rangle_\alpha \in (0,1]$ we have by lemma 3.1.

$$\sharp\{j \in \{1,\ldots,n\}|d_j = d\} = \sum_{i=0}^{n-1}\chi_{\left(\frac{1}{\alpha^d},\frac{1}{\alpha^{d-1}}\right]}(T^i(x)),$$

for all $d \in \mathbb{N}$. Fix $d \in \mathbb{N}$. By proposition 3.1 and Birkhoffs ergodic theorem we have for almost all $x = \langle d_i \rangle_\alpha \in (0,1]$

$$\mathfrak{f}_{d,\alpha}(x) = \lim_{n\to\infty}\frac{\sharp\{j \in \{1,\ldots,n\}|d_j = d\}}{n} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\chi_{\left(\frac{1}{\alpha^d},\frac{1}{\alpha^{d-1}}\right]}(T^i(x))$$

$$= \int_0^1 \chi_{\left(\frac{1}{\alpha^d},\frac{1}{\alpha^{d-1}}\right]}(x)dx = (\alpha-1)/\alpha^d.$$

The result follows since the countable intersection of sets with full measure has full measure. □

From this theorem it follows that the set of numbers with a bounded sequence of digits in their α-expansion has Lebesgue measure zero. We will see in the next section that the Hausdorff dimension of this set is one.

In our second theorem on the distribution of digits in α-expansion we consider the asymptotic arithmetic and geometric mean of the digits.

Theorem 3.2 *Let $\alpha > 1$. For almost all $x = \langle d_i \rangle_\alpha \in (0,1]$ we have*

$$\lim_{n\to\infty} \frac{1}{n}(d_1 + d_2 \cdots + d_n) = \frac{\alpha}{\alpha - 1}$$

and

$$\lim_{n\to\infty} \sqrt[n]{d_1 d_2 \dots d_n} = \prod_{d=1}^{\infty} \sqrt[\alpha^d]{d^{\alpha-1}}.$$

Proof. Let

$$f(x) = \sum_{d=1}^{\infty} d\chi_{\left(\frac{1}{\alpha^d}, \frac{1}{\alpha^{d-1}}\right]}(x).$$

By lemma 3.1 we have $f(T^{i-1}(x)) = d_i$ for all $i \in \mathbb{N}$, where $x = \langle d_i \rangle_\alpha$. Applying Birkhoff's ergodic theorem we obtain

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} d_i = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(T^{i-1}(x)) = \int_0^1 f(x)dx$$

$$= \sum_{d=1}^{\infty} d\frac{\alpha - 1}{\alpha^d} = \frac{\alpha}{\alpha - 1}$$

for almost all $x = \langle d_i \rangle_\alpha \in (0,1]$. Now let

$$f(x) = \sum_{d=1}^{\infty} \log(d)\chi_{\left(\frac{1}{\alpha^d}, \frac{1}{\alpha^{d-1}}\right]}(x).$$

By lemma 3.1 we here have $f(T^{i-1}(x)) = \log(d_i)$ for all $i \in \mathbb{N}$. Applying Birkhoff's ergodic theorem again we obtain

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log(d_i) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(T^{i-1}(x)) = \int_0^1 f(x)dx$$

$$= \sum_{d=1}^{\infty} \log(d)\frac{\alpha - 1}{\alpha^d}$$

for almost all $x = \langle d_i \rangle_\alpha \in (0,1]$. Applying the exponential on both sides gives the result on the geometric mean. □

Consider the function

$$G(x) = \prod_{d=1}^{\infty} \sqrt[\alpha^x]{d^{x-1}}.$$

$G(2) = 1.6616879496\dots$ is known as Somos' constant, see page 446 of [15]. In definition 2 of [30] the numbers $\sqrt[x-1]{G(x)}$ are called generalized Somos' constants. We would prefer to call $G : (0,\infty) \to \mathbb{R}$ the Somos' function. We display an approximation of this function in figure 1. We like to remark that from theorem 8 in [30] we get a relationship between G and the generalized-Euler-constant function γ:

$$G(x) = \frac{x}{x-1} e^{-\gamma(1/x)/x},$$

where

$$\gamma(x) = \sum_{i=1}^{\infty} x^{i-1} \left(\frac{1}{i} - \log(\frac{i+1}{i}) \right).$$

The proof of the formula is just a straightforward calculation.

4 Dimension theoretical aspects

For a proper subset $D \subset \mathbb{N}$ and $\alpha > 1$ let us consider the set of numbers which have only digits in D in their α-expansion,

$$I_\alpha(D) = \{\langle d_i \rangle_\alpha | d_i \in D\}.$$

This is a Cantor set hence its Hausdorff dimension $\dim_H I_\alpha(D)$ is of interest. The reader who is not familiar with basic dimension theory should consider our essential [28] or [13]. Applying well known results in dimension theory we obtain:

Theorem 4.1 *Let $D \subset \mathbb{N}$, $\alpha > 1$ and $h \geq 0$ be the unique solution of*

$$\sum_{i \in D} (\frac{\alpha - 1}{\alpha^i})^h = 1.$$

Than $\dim_H I_\alpha(D) = h$.

Proof. Let $\tilde{I}_\alpha(D) = I_\alpha(D) \cup \{0, 0\}$, adding a point does not change Hausdorff dimension. $\tilde{I}_\alpha(D)$ is the attractor of the iterated function system $([0,1], \{T_i | i \in D)$, since

$$\tilde{I}_\alpha(D) = \bigcup_{i \in D} T_i(\tilde{I}_\alpha(D)).$$

This iterated function system fulfills the open set condition since

$$T_i((0,1)) \cap T_j((0,1)) = \emptyset$$

if $i \neq j$. The dimension formula in our result is the classical Moran formula. If A is finite the result directly follows from the work of Moran [23]. If A is infinite it follows from theory of infinite iterated function systems see theorem 3.11 of [14] or [24]. □

For $D_n = \{1, \ldots, n\}$ we obtain $\dim_H I_\alpha(D_n) = h(n)$ where $h(n) > 0$ is given by

$$(1 - \alpha^{-h(n)n}) \frac{(\alpha - 1)^{h(n)}}{\alpha^{h(n)} - 1} = 1.$$

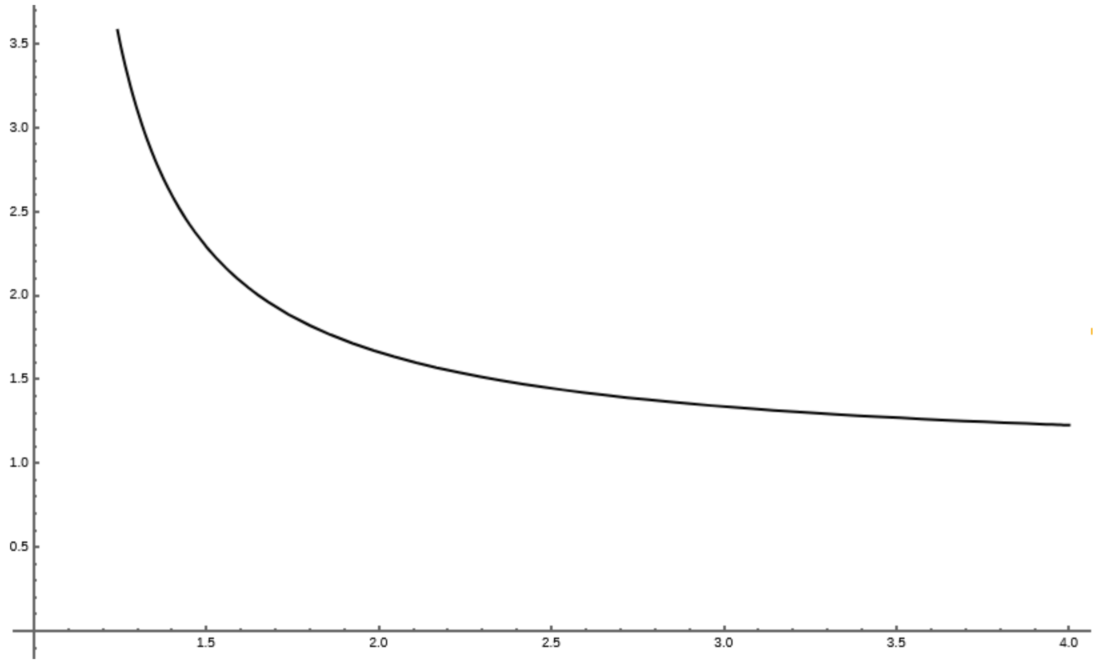

Figure 1: The Somos' function G

Note that $h(n) \to \infty$ for $n \to \infty$ hence

$$\dim_H \{\langle d_i \rangle_\alpha | (d_i) \text{ is bounded}\} = \dim_H \bigcup_{n=1}^{\infty} I_\alpha(D_n) = 1,$$

although the Lebesgue measure of this set is zero.

Next we study the set of real numbers with prescribed frequencies of digits in their α-expansion. We need some notations. Let $\mathfrak{p} = (p_1, \dots, p_n) \in [0,1)^n$ be a probability distribution on $\{1, \dots, n\}$ that means

$$\sum_{i=1}^{n} p_i = 1.$$

The expected value of $\mathfrak{p}$ is

$$E(\mathfrak{p}) = \sum_{i=1}^{n} i p_i$$

and the entropy is given by

$$H(\mathfrak{p}) = -\sum_{i=1}^{n} p_i \log(p_i).$$

We are interested in the set of real numbers with frequency p_d of the digit d in their α-expansion;

$$\mathfrak{F}_\alpha(\mathfrak{p}) = \{x \in (0,1] | \mathfrak{f}_{d,\alpha}(x) = p_d,\ d = 1, \dots, n\}.$$

These are fractal sets and we will proof the following dimension formula:

Theorem 4.2 *For all $\alpha > 1$ and all probability distribution $\mathfrak{p} \in [0,1)^n$ on $\{1,\ldots,n\}$ we have*

$$\dim_H \mathfrak{F}_\alpha(\mathfrak{p}) = \frac{H(\mathfrak{p})}{\log(\alpha)E(\mathfrak{p}) - \log(\alpha-1)}.$$

Proof. Let P be the probability measure on $\mathbb{N}^{\mathbb{N}}$ which is the product of the probability distribution $\mathfrak{p} = (p_1,\ldots,p_n,0,0,\ldots)$ on $\mathbb{N}$. We project P to a probability measure μ on $(0,1]$ using π,

$$\mu = \pi(P) = P \circ \pi^{-1}.$$

Be the law of large numbers for P-almost all sequences (d_i) the frequency of a digit d is given by p_d, hence $\mu(\mathfrak{F}_\alpha(\mathfrak{p})) = 1$.
The intervals

$$I_{d_1,\ldots,d_k} = T_{d_k} \circ \cdots \circ T_{d_1}((0,1])$$

have length and measure given by

$$|I_{d_1,\ldots,d_k}| = \frac{(\alpha-1)^k}{\alpha^{d_1+\cdots+d_k}}, \quad \mu(I_{d_k,\ldots,d_1}) = \prod_{j=1}^{k} p_{d_j},$$

compare with section 2.1. For $x = \langle d_i \rangle_\alpha \in (0,1]$ let $I_{d_1,\ldots,d_k}(x)$ the interval containing x and

$$\sharp_d(x|k) = \sharp\{j | d_j = d, \quad j = 1,\ldots,k\}.$$

For the length of the intervals we obtain

$$\log(|I_{d_1\ldots d_k}(x)|) = \log(\frac{(\alpha-1)^k}{\alpha^{d_1+\cdots+d_k}}) = k\log(\alpha-1) - \log(\alpha)\sum_{d=1}^{n} \sharp_d(x|k)d$$

and for the measure of the intervals we get

$$\log(\mu(I_{d_1\ldots d_k})(x)) = \log(\prod_{j=1}^{k} p_{d_j}) = \sum_{d=1}^{n} \sharp_d(x|k)\log p_d.$$

If $x \in \mathfrak{F}_\alpha(\mathfrak{p})$, we have

$$\lim_{k\to\infty} \frac{\sharp_d(x|k)}{k} = p_d$$

for all $d \in \mathbb{N}$. Hence

$$\lim_{k\longmapsto\infty} \frac{1}{k} \log \frac{\mu(I_{n_1\ldots n_k}(x))}{|I_{n_1\ldots n_k}(x)|^s}$$
$$= -H(\mathfrak{p}) + s(\log(\alpha)E(\mathfrak{p}) - \log(\alpha-1)).$$

This implies

$$\lim_{k\longmapsto\infty} \frac{\mu(I_{n_1\ldots n_k}(x))}{|I_{n_1\ldots n_k}(x)|^s} = \{ \begin{array}{cc} 0 & s < h \\ \infty & s > h \end{array},$$

where

$$h = \frac{H(\mathfrak{p})}{\log(\alpha)E(\mathfrak{p}) - \log(\alpha - 1)}.$$

By the local mass distribution principle, see proposition 4.9 of [13], we have $\mathfrak{H}^s(\mathfrak{F}_\alpha(\mathfrak{p})) = \infty$ for $s < h$ and $\mathfrak{H}^s(\mathfrak{F}_\alpha(\mathfrak{p})) = 0$ for $s > h$, where $\mathfrak{H}^s$ is the s-dimensional Hausdorff measure. This implies $\dim_H(\mathfrak{F}_\alpha(\mathfrak{p})) = h$. □

5 The base-change-transformation

For fixed real numbers $\alpha, \beta > 1$ we define a base-change-transformation by $f : [0,1] \to [0,1]$ by $f(x) = f(\langle d_i \rangle_\alpha) = \langle d_i \rangle_\beta$, that is

$$f(x) = f(\sum_{i=1}^{\infty} (\alpha - 1)^{i-1} \alpha^{-(d_1 + \cdots + d_i)}) = \sum_{i=1}^{\infty} (\beta - 1)^{i-1} \beta^{-(d_1 + \cdots + d_i)}$$

for $x \neq 0$ and $f(0) = 0$. Obviously we have the functional equation

$$f(x/\alpha) = f(x)/\beta$$

for all $x \in [0,1]$. If $\alpha = \beta$ the map f is the identity on $[0,1]$. In general we have:

Theorem 5.1 $f : [0,1] \to [0,1]$ *is strictly increasing and continuous.*

Proof. Let us introduce some notations. For $(d_i) \in \mathbb{N}^{\mathbb{N}}$ we set

$$\Xi(d_i) = \min\{k | d_i = 1 \forall i > k\}$$

if the minimum exists and $\Xi(d_i) = \infty$ if this is not the case. For two different sequences $(d_i), (g_i) \in \mathbb{N}^{\mathbb{N}}$ we set

$$|(d_i) \wedge (g_i)| = \min\{i | d_i \neq g_i\}.$$

Let $x, y \in (0,1]$ with $x = \langle d_i \rangle_\alpha$ and $y = \langle g_i \rangle_\alpha$. If $x < y$, we have $d_{|(d_i) \wedge (g_i)|} > g_{|(d_i) \wedge (g_i)|}$. Hence $f(x) = \langle d_i \rangle_\beta < \langle g_i \rangle_\beta = f(y)$. If $x > 0$ obviously $f(x) > f(0) = 0$. Hence f is strictly increasing.

Let $(x_n) = \langle d_i(n) \rangle_\alpha \in (0,1]^{\mathbb{N}}$ be a sequence. If $\lim_{n\to\infty} x_n = 0$, we have $\lim_{n\to\infty} d_i(x_n) = \infty$ for all $i \in \mathbb{N}$. Hence

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \langle d_i(n) \rangle_\beta = 0 = f(0)$$

and f is continuous in $x = 0$.

Now let $x = \langle d_i \rangle_\alpha \in (0,1]$ and $\lim_{n\to\infty} x_n = x$. If $\Xi(d_i) = \infty$, we have $\lim_{n\to\infty} |(d_i(n)) \wedge (d_i)| = \infty$ hence

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \langle d_i(n) \rangle_\beta = \langle d_i \rangle_\beta = f(x).$$

Now let $\Xi(d_i) = k$ and let (n_j) be a sequence of natural numbers with $\lim_{k\to\infty} n_j = 0$. There are to possibilities. Either $\lim_{j\to\infty} |(d_i(n_j)) \wedge (d_i)| = \infty$ or $d_k(n_j) = d_k - 1$ for j sufficient large and $\lim_{j\to\infty} d_i(n_j) = \infty$ for all $i > k$. In both cases

$$\lim_{j\to\infty} f(x_{n_j}) = \lim_{j\to\infty} \langle d_i(n_j)\rangle_\beta = \langle d_i\rangle_\beta = f(x).$$

Hence f is continuous for all $x \in (0,1]$. □

Now we have a look on the structure of the graph of f,

$$F = \{(x, f(x)) | x \in [0,1]\} \subseteq [0,1] \times [0,1],$$

$$= \{0,0\} \cup \{\sum_{i=1}^{\infty}(\alpha-1)^{i-1}\alpha^{-(d_1+\cdots+d_i)}, \sum_{i=1}^{\infty}(\beta-1)^{-1}\beta^{-(d_1+\cdots+d_i)}) | (d_i) \in \mathbb{N}^{\mathbb{N}}\}.$$

We display an approximation of F for $(\alpha, \beta) = (3,2), (3,10)$ in figure 2. In addition to theorem 5.1 we obtain:

Theorem 5.2 *F is an infinite self-affine curve with Hausdorff dimension one.*

Proof. Consider affine contractions $G_i : [0,1] \times [0,1] \to [0,1] \times [0,1]$, given by

$$G_i(x,y) = (\frac{\alpha-1}{\alpha^i}x + \frac{1}{\alpha^i}, \frac{\beta-1}{\beta^i}y + \frac{1}{\beta^i})$$

for $i \in \mathbb{N}$. We have

$$\bigcup_{i=1}^{\infty} G_i(F) = F,$$

which means that F is infinite self-affine. We have $\dim_H F \geq 1$ since the projection of F to the first coordinate axis is the whole interval and projections do not increase Hausdorff dimension. Now consider the finite iterated function system, given by $([0,1]^2, \{G_1, G_\infty\})$, where G_∞ is given by

$$G_\infty(x,y) = (\frac{1}{\alpha}x, \frac{1}{\beta}y).$$

Since $G_i([0,1]^2) \subseteq G_\infty([0,1]^2)$ for all $i \geq 2$ the attractor Λ of this finit iterated function system contains F. But it follows from the theory of finit self-affine sets, that $\dim_H \Lambda = 1$, see proposition 10.2.6 in [3]. Hence $\dim_H F = 1$. □

At the end we like to include a result on the upper derivative $\overline{D}_x f$ resp. lower derivative $\underline{D}_x f$ of the base-change-transformation f.

Theorem 5.3 *For $\alpha, \beta > 1$ with $\alpha \neq \beta$ let f be the corresponding base-change-transformation. For almost all $x \in (0,1]$ we have $\overline{D}_x f = \infty$ and $\underline{D}_x f = 0$.*

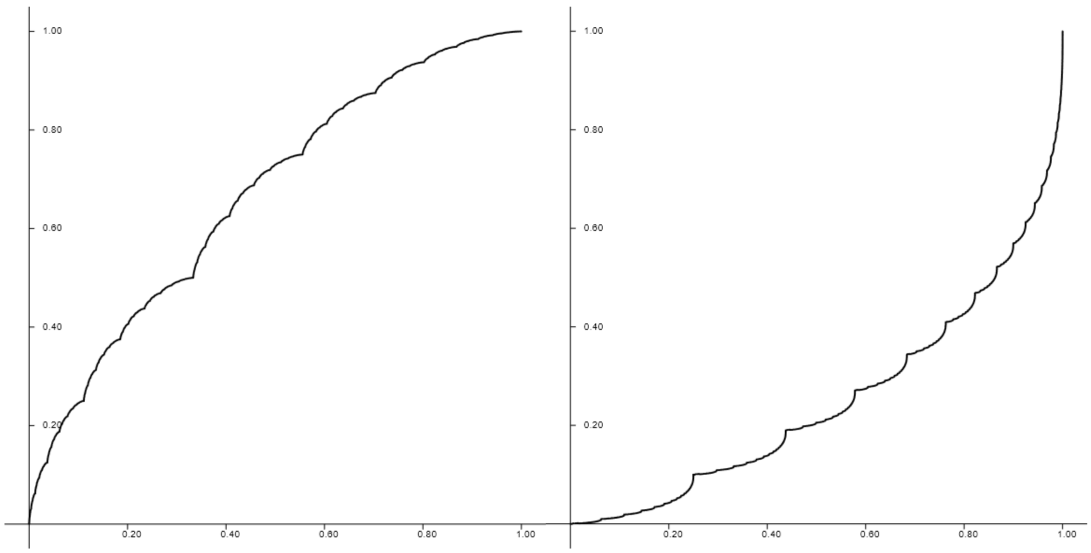

Figure 2: An approximation of the base-change-transformation for $(\alpha, \beta) = (3, 2), (3, 10)$.

Proof. Consider $f(x) = \langle d_i \rangle_\beta \in (0, 1]$ with

$$\lim_{n\to\infty} \frac{1}{n}(d_1 + d_2 \cdots + d_n) = \frac{\beta}{\beta - 1}.$$

From theorem 3.2. we know that the set of $x = \langle d_i \rangle_\alpha \in (0, 1]$ with these property has full measure. Define

$$x_n = T_{d_n} \circ \cdots \circ T_{d_1}(0) \text{ and } y_n = T_{d_n} \circ \cdots \circ T_{d_1}(x)(1)$$

We obviously have $x_n < x < y_n$ and $\lim_{n\to\infty} x_n = x$ as well as $\lim_{n\to\infty} y_n = x$. Furthermore we obtain

$$\lim_{n\to\infty} \frac{f(y_n) - f(x_n)}{y_n - x_n} = \lim_{n\to\infty} \frac{(\beta-1)^n \beta^{-d_1 - \cdots - d_n}}{(\alpha-1)^n \alpha^{-d_1-\cdots-d_n}} = \lim_{n\to\infty} \left(\frac{(\beta-1)\alpha^{(d_1+\cdots+d_n)/n}}{(\alpha-1)\beta^{(d_1+\cdots+d_n)/n}} \right)^n = \infty.$$

since

$$\frac{(\beta-1)\alpha^{\beta/(\beta-1)}}{(\alpha-1)\beta^{\beta/(\beta-1)}} > 1$$

for $\alpha, \beta > 1$ with $\alpha \neq \beta$. Hence $\overline{D}_x f = \infty$ for almost all $x \in (0, 1]$. Furthermore this implies $\underline{D}_x f^{-1} = 0$ for almost all $x \in (0, 1]$, where f^{-1} is the base-change-transformation from β to α. Interchanging α and β the stated result follows for (α, β) and (β, α) since the intersection of sets with full measure has full measure.

□

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