On the dimension of certain sets araising in the base two expansion

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Abstract

We show that for the base two expansion

$$x=\sum_{i=1}^{\infty}2^{-(d_1(x)+d_2(x)+\cdots+d_i(x))}$$

with $x\in(0,1]$ and $d_i(x)\in\mathbb{N}$ the set $A=\{x|\lim_{i\to\infty}d_i(x)=\infty\}$ has Hausdorff dimension zero, this is opposed to a result on the continued fraction expansion, here A has Hausdorff dimension 1/2, see [2]. Furthermore we construct subsets of $B=\{x|\limsup_{i\to\infty}d_i(x)=\infty\}$ which have Hausdorff dimension one and find a dimension spectrum in set B.

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1 Introduction and main result

There are various ways to expand real numbers in infinite sequences of natural numbers. For instance there are continued fraction expansions, the Engel expansion, the Lüroth expansion, the Sylvester expansion and the base two expansion we introduced in [10]. These expansions play a prominent role in the theory of real numbers and are closely connected with fractal geometry, dynamical systems and measure theory. Since the work of Jarnick [7] from the 1920th the dimension theory of such expansions is consider and became one main issue of metric number theory. We like to remind the reader about some results concerning the classical continued fraction expansion

$$x=\cfrac{1}{a_1(x)+\cfrac{1}{a_2(x)+\cfrac{1}{a_3(x)+\cfrac{1}{\ddots}}}},$$

of an irrational number $x\in[0,1]\backslash\mathbb{Q}$ with $a_i(x)\in\mathbb{N}$. Jarnick found upper and lower bounds on the Hausdorff dimension of $\{x\in[0,1]\backslash\mathbb{Q}|1\le a_i(x)\le M\}$, which imply

$$\dim_H\{x\in[0,1]\backslash\mathbb{Q}|a_i(x)\text{ is bounded}\}=1,$$

although the Lebesgue measure of this set is known to be zero.[1] From Good's result in the 1950th [2] we know that

$$\dim_H\{x \in [0,1]\backslash\mathbb{Q} | \lim_{i\to\infty} a_i(x) = \infty\} = 1/2.$$

Furthermore the set

$$\{x \in [0,1]\backslash\mathbb{Q} | \limsup_{i\to\infty} a_i(x) = \infty\}$$

has positive Lebesgue measure and hence Hausdorff dimension one. Today we have a complete multiracial analysis of sets of continued fractions with given growth rate of digits, see [4, 5, 8] and multifractal results for the Engel, the Lüroth and the Sylvester expansion as well, see [13, 1, 15].
We consider here the base two expansion

$$x = \sum_{i=1}^{\infty} 2^{-(d_1(x)+d_2(x)+\cdots+d_i(x))}$$

with $d_i(x) \in \mathbb{N}$ of a real number $x \in (0,1]$. The digits are given by

$$d_i(x) = \lceil -\log_2(T^{i-1}(x)) \rceil,$$

where $Tx = 2^n x - 1$ for $x \in (1/2^n, 1/2^{n-1}]$ with $n \in \mathbb{N}$, see figure 1. It is easy to show that the expansion is unique. In [10] we found the Hausdorff dimension of $\{x \in (0,1] | 1 \le d_i(x) \le M\}$ (see proposition 3.1 below) and

$$\dim_H\{x \in (0,1] | d_i(x) \text{ is bounded}\} = 1,$$

while the Lebesgue measure of the set is zero. In the next section we prove:

Theorem 1.1 *We have*

$$\dim_H\{x \in (0,1] | \lim_{i\to\infty} d_i(x) = \infty\} = 0$$

This result is opposed to the result of Good aforementioned. In [12] we have show that the Lebesgue measure the ergodic with respect to T which implies:

Theorem 1.2 *The set*

$$B = \{x \in [0,1]\backslash\mathbb{Q} | \limsup_{i\to\infty} d_i(x) = \infty\}$$

has positive Lebesgue measure.

[1]The reader who is not familiar with Lebesgue measure and Hausdorff dimension should read our essential [11] or the excellent book [3]

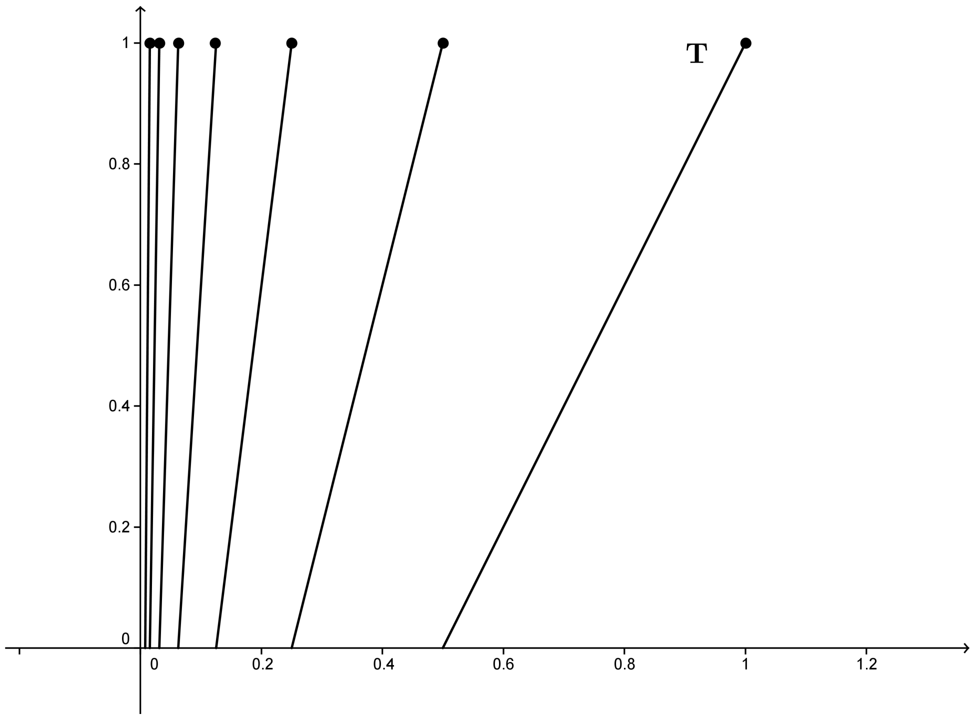

Figure 1: The map T

In section three we are interested in subsets of B with full dimension. Especially we will show that

$$\dim_H\{x \in (0,1] | d_{2^i}(x) = i \ \forall i \in \mathbb{N}\} = 1.$$

In section four we find a dimension spectrum in B, for instance we will find a sequence $k(i)$ of natural numbers such that

$$\dim_H\{x \in (0,1] | d_{k(i)}(x) = i\mu \ \forall i \in \mathbb{N}\} = \log_2(\gamma_\mu),$$

where $\gamma_\mu \in (1/2, 1)$ is the solution of $x^{\mu+1} + x - 1 = 0$.

2 Sequences of digits tending to ∞

We show in this section that the set

$$A = \{x \in (0,1] | \lim_{i\to\infty} d_i(x) = \infty\}$$

has Hausdorff dimension zero. To this end we define a set

$$A_M = \{x \in (0,1] | d_i(x) \geq M \ \forall i \in \mathbb{N}\}$$

for an integer $M \geq 1$. Obviously $\dim_H A_1 = 1$, moreover we obtain:

Proposition 2.1 *For all* $M \geq 2$ *we have* $\dim_H A_M = \log_2(\alpha_M)$, *where* $\alpha_M \in (1,2)$ *is the solution of* $x^M - x - 1 = 0$.

Proof. Set $D = \log_2(\alpha_M)$. We have

$$\sum_{i=M}^{\infty} 2^{-Di} = \frac{2^{-DM}}{1-2^{-D}} = \frac{\alpha_M^{-M}}{1-\alpha_M^{-1}} = \frac{1}{\alpha_M^M - \alpha_M} = 1.$$

By theorem 2.1 of [10] this implies $\dim_H A_M = D$. □

Note that $\lim_{M\to\infty} \alpha_M = 1$ and hence $\lim_{M\to\infty} \dim_H A_M = 0$. Therefore the following proposition implies $\dim_H A = 0$.

Proposition 2.2 *For all* $M \geq 1$ *we have* $\dim_H A \leq \dim_H A_M$.

Proof. Fix $M \geq 1$. For all integers $N \geq 1$ we definite

$$A_{M,N} = \{x \in (0,1] | d_i(x) \geq M \ \forall i > N\}.$$

We have

$$A_{M,N} = \bigcup_{s_1,\dots,s_N \in \mathbb{N}} \{x \in (0,1] | d_i(x) \geq M \ \forall i > N, \quad d_i(x) = s_i \ \forall i \leq N\}.$$

Let $T_i(x) = (x+1)/2^i$

$$T_{s_1} \circ \cdots \circ T_{s_n}(A_M) = \{x \in (0,1] | d_i(x) \geq M \ \forall i > N, \quad d_i(x) = s_i \ \forall i \leq N\}.$$

Since a linear bijection does not change Hausdorff dimension, all these sets have dimension $\dim_H A_M$. Since Hausdorff dimension is countable stable, we have $\dim_H A_{M,N} = \dim_H A_M$.
For $x \in A$ we have $\lim_{i\to\infty} d_i(x) = \infty$, hence there is a N such that $d_i(x) \geq M \ \forall i > N$, hence $x \in A_{M,N}$ for some N. This means

$$A \subseteq \bigcup_{N=1}^{\infty} A_{M,N}.$$

Again by countable stability the Hausdorff dimension of the union is $\dim_H A_M$ and $\dim_H A \leq \dim_H A_M$ follows by monotony of dimension. □

3 Set of full dimension

We construct here subsets of

$$B = \{x \in (0,1] | \limsup_{i\to\infty} d_i(x) = \infty\}$$

which have Hausdorff dimension one. We will use the following result on

$$B_M = \{x \in (0,1] | d_i(x) \le M \ \forall i \in \mathbb{N}\},$$

given in [10].

Proposition 3.1 *For all $M \ge 2$ we have $\dim_H B_M = \log_2(\beta_M)$, where $\beta_M \in (1,2)$ is the solution of $x^M - x^{M-1} - \dots x^2 - x - 1 = 0$.*

Note that $\lim_{M\to\infty} \beta_M = 2$ and hence $\lim_{M\to\infty} \dim_H A_M = 1$.
Now let $I \subseteq \mathbb{N}$ be an infinit set of digits with infinite compliment and $f : \mathbb{N} \to \mathbb{N}$ an arbitrary map. We set

$$B_M(I,f) = \{x \in (0,1] | d_i(x) = f(i) \ \forall i \in I, \ d_i(x) \le M \ \forall i \in \mathbb{N}\backslash I\}.$$

The digits of x in this set are given by $f(i)$ for an Index $i \in I$ and are bounded by M for other indices. For $n \in \mathbb{N}$ choose $k(n)$ such that the cardinality of $(\mathbb{N}\backslash I) \cap \{1,2,\dots,k(n)\}$ is n. Moreover set

$$\lambda_n = \lambda_n(I,f) = \sum_{j \in I \cap \{1,2,\dots,k(n)\}} f(j) \quad \text{and} \quad \mu(I,f) = \limsup_{n\to\infty} \lambda_n/n.$$

With this notations we have the following result:

Proposition 3.2 *Let $I \subseteq \mathbb{N}$ be an infinite set with infinite complement and $f : \mathbb{N} \to \mathbb{N}$ by a function such that $\mu(I,f) = 0$. Then $\dim_H B_M(I,f) = dim_H B_M$ for all $M \ge 2$.*

Proof. The set B_M is the attractor of the iterated function system $T_i(x) = (x+1)/2^i$ $1 \le i \le M$;

$$B_M = \bigcup_{i=1}^{M} T_i(B_M).$$

It is well known that for such self-similar sets there is a self-similar probability measure ν on the set with full Hausdorff Dimension; $\dim_H \nu = \dim_H B_M$. This measure is given by

$$\nu(\{x \in (0,1] | d_i(x) = j\}) = 2^{-j \dim_H B_M}$$

for all $j \in \{1,\dots,M\}$ and all $i \in \mathbb{N}$. For $j > M$ the measure is zero. See theorem 9.3 of [3]. We project this measure to a probability measure $\hat{\nu}$ on $B_M(I,f)$ by

$$\hat{\nu}(\{x \in (0,1] | d_i(x) = j\}) = \nu(\{x \in (0,1] | d_i(x) = j\})$$

for $i \in I\backslash\mathbb{N}$ and

$$\hat{\nu}(\{x \in (0,1] | d_i(x) = f(i)\} = 1, \quad \hat{\nu}(\{x \in (0,1] | d_i(x) \ne f(i)\} = 0$$

for $i \in I$. For a sequence $(s_1, \ldots, s_n) \in \mathbb{N}^n$ let

$$V(s_1, \ldots, s_n) = \{x \in (0,1] | d_i(x) = s_i, i = 1, \ldots, n\}$$

and

$$\hat{V}(s_1, \ldots, s_n) =$$

$$\{x \in (0,1] | d_i(x) = s_i \forall i \in (\mathbb{N} \backslash I) \cap \{1, 2, \ldots, k(n)\},\ d_i(x) = f(i) \forall i \in I \cap \{1, 2, \ldots, k(n)\}\}.$$

The first set is an interval of length $2^{-(s_1+\cdots+s_n)}$ and the second set is an interval of length $2^{-(s_1+\cdots+s_n+\lambda_n)}$. Moreover by definition of $\hat{\nu}$ we have

$$\nu(V(s_1, \ldots, s_n)) = \hat{\nu}(\hat{V}(s_1, \ldots, s_n)).$$

Recall that by definition of $d_i(x)$ for all $x \in (0,1]$, the interval $V(d_1(x), \ldots, d_n(x))$ contains x. The self-similar measure ν is known to be exact dimensional, see theorem 2.12 of [6]. This means

$$\lim_{n\to\infty} \frac{\log_2(\nu(V(d_1(x), \ldots, d_n(x))))}{\log_2(|V(d_1(x), \ldots, d_n(x))|)} = \dim_H B_M$$

for ν-almost all $x \in B_M$. Hence

$$\lim_{n\to\infty} \frac{\log_2(\hat{\nu}(\hat{V}(d_1(x), \ldots, d_n(x))))}{\log_2(|\hat{V}(d_1(x), \ldots, d_n(x))|)}$$

$$= \lim_{n\to\infty} \frac{\log_2(\nu(V(d_1(x), \ldots, d_n(x))))}{-\log(2)(d_1(x) + \cdots + d_n(x) + \lambda_n)}$$

$$= \dim_H B_M \lim_{n\to\infty} \frac{1}{1 + \lambda_n/(d_1(x) + \cdots + d_n(x))} = \dim_H B_M$$

for $\hat{\nu}$-almost all $x \in B_M(I, f)$. In the last equation we used the assumption $\mu(I, f) = 0$. By theorem 4.4 of [14] this implies $\dim_H \hat{\nu} = \dim_H B_M$, hence $\dim_H B_M(I, f) = \dim_H B_M$. □

Let us now consider the set

$$B(I, f) = \{x \in (0,1] | d_i(x) = f(i)\ \forall i \in I\}.$$

Combining proposition 3.1 and 3.2 we easily obtain:

Proposition 3.3 *For all infinit sets $I \subseteq \mathbb{N}$ with infinite complement and $f : \mathbb{N} \to \mathbb{N}$ with $\mu(I, f) = 0$ we have $\dim_H B(I, f) = 1$.*

Proof. Using the countable stability of Hausdorf dimension we have

$$\dim_H B(I, f) = \dim_H \bigcup_{M=1}^{\infty} B_M(I, f) = \sup\{\dim_H B_M(I, f) | M \geq 1\}$$

$$= \sup\{\dim_H B_M | M \geq 1\} = 1.$$

□

As an example let $I = \{2^n | n \in \mathbb{N}\}$ and $f(x) = \log_2(x)$. We have $\mu(I, f) = 0$ and hence $\dim_H B(I, f) = 1$.

4 A dimension spectrum

We will show here that there is a kind of dimension spectrum inside B. Using the notations of last section, we will show:

Proposition 4.1 *Let $I \subseteq \mathbb{N}$ be an infinite set with infinite complement and $f : \mathbb{N} \to \mathbb{N}$ by a function such that*

$$\liminf_{n\to\infty} |\lambda_n(I, f) - n\mu(I, f)| < \infty.$$

We have $\dim_H B_M(I, f) = -\log_2(\gamma_{\mu,M})$ for all $M \geq 2$, where $\gamma_{\mu,M} \in (1/2, 1)$ is the solution of

$$x^\mu(x^M + x^{M-1} + \ldots x^2 + x) = 1$$

and $\dim_H B(I, f) = -\log_2(\gamma_\mu)$, where $\gamma_\mu \in (1/2, 1)$ is the solution of

$$x^{\mu+1} + x - 1 = 0.$$

Proof. We fix I, f and let $\lambda_n = \lambda_n(I, f)$ and $\mu = \mu(I, f)$. Moreover we fix $M \geq 2$ and let $D = -\log_2(\gamma_{\mu,M})$. By our assumption there is a growing sequence of positive integers $n(k)$ and a constant $c \geq 0$, such that

$$|\lambda_{n(k)} - n(k)\mu| \leq c$$

for all $k \in \mathbb{N}$. We want to estimate the D-dimensional Hausdorff measure of $B_M(I, f)$ at level $2^{-n(k)}$;

$$\mathfrak{H}^D_{2^{-n(k)}}(B_M(I, f)) = \inf\{\sum_{j=1}^{\infty} |C_j|^D \mid B_M(I, f) \subseteq \bigcup_{j=1}^{\infty} C_j,\ |C_j| \leq 2^{-n(k)}\}.$$

Using the sets $\hat{V}(s_1, \ldots, s_{n(k)})$, defined in the last section as a cover of $B_M(I, f)$, we have

$$\mathfrak{H}^D_{2^{-n(k)}}(B_M(I, f)) \leq \sum_{1\leq s_1,\ldots,s_{n(k)}\leq M} |\hat{V}(s_1, \ldots, s_{n(k)})|^D$$

$$= \sum_{1\leq s_1,\ldots,s_{n(k)}\leq M} 2^{-D(s_1+\cdots+s_{n(k)}+\lambda_{n(k)})} = 2^{-D\lambda_{n(k)}}(2^{-D} + 2^{-2D} + \cdots + 2^{-MD})^{n(k)}$$

$$\leq 2^{-Dn(k)\mu+Dc}(2^{-D}+2^{-2D}+\cdots+2^{-MD})^{n(k)} = 2^{-Dc}(2^{-D\mu}(2^{-D}+2^{-2D}+\cdots+2^{-MD}))^{n(k)} = 2^{-Dc}.$$

In the last equation we used the definition of D. Taking the limit $k \to \infty$ and using the monotony of Hausdorff measure with respect to the level, we obtain

$$\mathfrak{H}^D(B_M(I,f)) = \lim_{\epsilon \to 0} \mathfrak{H}^D_\epsilon(B_M(I,f)) < \infty.$$

By the definition of Hausdorff dimension this implies $\dim_H B_M(I,f) \le D$. For the lower bound on the dimension we will use the probability measure $\bar{\nu}$ on $B_M(I,f)$, given by

$$\bar{\nu}(\{x \in (0,1] | d_i(x) = j\}) = 2^{-D(\mu+j)}$$

for $i \in I \backslash \mathbb{N}$ and

$$\bar{\nu}(\{x \in (0,1] | d_i(x) = f(i)\} = 1, \quad \bar{\nu}(\{x \in (0,1] | d_i(x) \neq f(i)\} = 0$$

for $i \in I$. We have

$$\begin{aligned} &\lim_{n\to\infty} \frac{\log_2(\bar{\nu}(\hat{V}(d_1(x),\dots,d_n(x))))}{\log_2(|\hat{V}(d_1(x),\dots,d_n(x))|)} \\ &= \lim_{n\to\infty} \frac{\log_2(2^{-(D(\mu+d_1(x)+\mu)+\cdots+D(\mu+d_n(x)))})}{\log_2(2^{-(d_1(x)+\cdots+d_n(x)+\lambda_n)})} \\ &= D \lim_{n\to\infty} \frac{d_1(x)+\cdots+d_n(x)+n\mu}{d_1(x)+\cdots+d_n(x)+\lambda_n} = D. \end{aligned}$$

Again by theorem 4.4 of [14] this implies $\dim_H \bar{\nu} = D$ and $\dim_H B_M(I,f) \ge D$, which completes the proof of the first part of the proposition. The second part follows immediately from the countable stability of Hausdorf dimension and $\lim_{M\to\infty} \gamma_{\mu,M} = \gamma_\mu$. □

Let us discuss one application of proposition 4.1. Let

$$I = \{\frac{i^2+3i-2}{2} | i \in \mathbb{N}\} \quad \text{and} \quad f(\frac{i^2+3i-2}{2}) = \mu i$$

with $\mu \in \mathbb{N}$. We have chosen I and f such that

$$\lambda_{(n^2+2)/2+j}(I,f) = \mu(n^2+n)/2 \quad \text{for} \quad j = 0,\dots,n-1$$

for all $n \in \mathbb{N}$. This implies

$$\lim_{n\to\infty} \frac{\lambda_n(I,f)}{n} = \mu \quad \text{and} \quad |\lambda_{(n^2+2)/2}(I,f) - \mu(n^2+n)/2| = 0.$$

Proposition 4.1 gives

$$\dim_H\{x \in (0,1] | d_{\frac{i^2+3i-2}{2}}(x) = \mu i, i \in \mathbb{N}\} = -\log_2(\gamma_\mu),$$

where $\gamma_\mu \in (1/2,1)$ is the solution of $x^{\mu+1} + x - 1 = 0$.

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