

# On the universality of Somos' constant

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## Abstract

We show that Somos' constant is universal in sense that is similar to the universality of the Khinchin constant. In addition we introduce generalized Somos' constants, which are universal in a similar sense.

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## 1 Introduction and main result

Let us first recall the Khinchin constant

$$K = \prod_{i=1}^{\infty} \left( 1 + \frac{1}{i(i+2)} \right)^{\log_2 i} = 2.6854520010 \dots$$

By the famous theorem of Khinchin [3] this constant is universal in the following sense: For almost all real numbers  $x$  the geometric mean of the entries of the continued fractions of  $x$  converges to  $K$ . We consider here Somos' constant

$$\sigma = \prod_{i=1}^{\infty} \sqrt[i]{i} = 1.6616879496 \dots,$$

which first appeared in [7] in the context of the quadratic recurrence  $g_n = ng_{n-1}^2$ , see also page 446 of [1]. In the recent past this constant raised some attention, see for instance [2, 4, 6]. We will show that the Somos' constant is universal in a sense that is similar to the universality of the Khinchin constant. In [5] we represent real numbers  $x \in (0, 1]$  in the form

$$x = \langle n_1, n_2, n_3, \dots \rangle := \sum_{k=1}^{\infty} 2^{-(n_1+n_2+\dots+n_k)}$$

with  $n_k \in \mathbb{N}$  and show that the representation is unique. Replacing the continued fraction representation by this representation, we obtain the universality of Somos' constant.

**Theorem 1.1** *For almost all  $x = \langle n_1, n_2, n_3, \dots \rangle \in (0, 1]$  we have*

$$\lim_{i \rightarrow \infty} \sqrt[i]{n_1 n_2 \dots n_i} = \sigma.$$

We will prove this theorem in the next section. In the last section we will introduce generalized Somos' constants, which are universal with respect to a modification of the representation used here.

## 2 Proof of the main result

Consider the map  $T : (0, 1] \rightarrow (0, 1]$ , given by  $T(x) = 2^i x - 1$  for  $x \in (1/2^i, 1/2^{i-1}]$  and  $i \in \mathbb{N}$ . The relation of this transformation to the expansion of real numbers, defined in the last section is given by

**Lemma 2.1** *Let  $x = \langle n_1, n_2, n_3, \dots \rangle \in (0, 1]$ . For all  $k \in \mathbb{N}$  we have  $T^{k-1}(x) \in (1/2^i, 1/2^{i-1}]$  if and only if  $n_k = i$ .*

**Proof.** Obviously  $T(\langle n_1, n_2, n_3, \dots \rangle) = \langle n_2, n_3, n_4, \dots \rangle$ . Since  $x \in (1/2^i, 1/2^{i-1}]$  if and only if  $n_1 = i$  the result follows immediately.  $\square$

To apply Birkhoff's ergodic theorem we prove:

**Proposition 2.1** *The Lebesgue measure  $\mathfrak{L}$  is ergodic with respect to  $T$ .*

**Proof.** For an open interval  $(a, b) \subseteq [0, 1]$  we have

$$\begin{aligned} \mathfrak{L}(T^{-1}((a, b))) &= \mathfrak{L}\left(\bigcup_{i=1}^{\infty} (a/2^i + 1/2^i, b/2^i + 1/2^i)\right) \\ &= \sum_{i=1}^{\infty} 2^{-k} \mathfrak{L}\left((a/2^i + 1/2^i, b/2^i + 1/2^i)\right) = \sum_{i=1}^{\infty} 2^{-i} (b - a) = b - a = \mathfrak{L}((a, b)). \end{aligned}$$

Hence  $\mathfrak{L}(T^{-1}(B)) = \mathfrak{L}(B)$  for all Borel sets  $B \subseteq (0, 1]$ , which means that  $\mathfrak{L}$  is invariant under  $T$ . Let  $B$  be a Borel set with  $\mathfrak{L}(B) < 1$ , which is invariant under  $T$ ; that is  $T(B) = B$ . Note that for all  $k \in \mathbb{N}$  the intervals of the form

$$I_{m_1, \dots, m_k} = \{\langle n_1, n_2, n_3, \dots \rangle | n_i = m_i \text{ for } i = 1, \dots, k\}$$

build a partition of  $(0, 1]$ , where the length of the partition elements is bounded by  $1/2^k$ . By Lebesgue's density theorem for every  $\epsilon > 0$  there is an interval  $I = I_{m_1, \dots, m_k}$  such that  $\mathfrak{L}(I \setminus B) \geq (1 - \epsilon)\mathfrak{L}(I)$ . Since  $T^k(I) = (0, 1]$  we have

$$\mathfrak{L}((0, 1] \setminus B) \geq \mathfrak{L}(T^k(I \setminus B)) \geq (1 - \epsilon)\mathfrak{L}(T^k(I)) = 1 - \epsilon.$$

Hence  $\mathfrak{L}(B) = 0$ . This proves that  $\mu$  is ergodic.  $\square$

Now we are prepared to prove Theorem 1.1. Let  $f(x) = \sum_{i=1}^{\infty} \log(i) \chi_{(1/2^i, 1/2^{i-1}]}(x)$ , where  $\chi$  is the characteristic function. By lemma 2.1 we have  $f(T^{k-1}(x)) = \log(n_k)$  for  $x = \langle n_1, n_2, n_3, \dots \rangle$ . Applying Birkhoff's ergodic theorem to  $T$  with the  $L^1$ -function  $f$ , we obtain

$$\lim_{i \rightarrow \infty} \frac{1}{i} \sum_{k=1}^i \log(n_k) = \lim_{i \rightarrow \infty} \frac{1}{i} \sum_{k=1}^i f(T^{k-1}(x)) = \int_0^1 f(x) dx$$

$$= \sum_{i=1}^{\infty} \log(i) 2^{-i}$$

for almost all  $x = \langle n_1, n_2, n_3, \dots \rangle \in (0, 1]$ . Taking the exponential gives the result.

### 3 A generalisation

Let  $b \geq 2$  be an integer. It is easy to show that a real numbers  $x \in (0, 1]$  has a unique representation in the form

$$x = \langle n_1, n_2, n_3, \dots \rangle_b := (b-1) \sum_{k=1}^{\infty} b^{-(n_1+n_2+\dots+n_k)}$$

with  $n_k \in \mathbb{N}$ , the argument can be found in [5]. Now consider the map  $T_b : (0, 1] \rightarrow (0, 1]$ , given by  $T_b(x) = b^i x - (b-1)$  for  $x \in ((b-1)b^{-i}, (b-1)b^{1-i}]$  and  $i \in \mathbb{N}$ . Using the argument in the last section with respect to  $T_b$  instead of  $T$  we obtain:

**Theorem 3.1** *For almost all  $x = \langle n_1, n_2, n_3, \dots \rangle_b \in (0, 1]$  we have*

$$\lim_{i \rightarrow \infty} \sqrt[i]{n_1 n_2 \dots n_i} = \prod_{i=1}^{\infty} \sqrt[i]{i^{b-1}} =: \sigma_b.$$

We like to call  $\sigma_b$  for  $b > 2$  a generalized Somos' constant. These constants are universal with respect to the base  $b$  representation  $\langle n_1, n_2, n_3, \dots \rangle_b$ . The generalization given here is slightly different from the generalization of Somos' constant studied in [8], which is not related to universality.<sup>1</sup>

We like to end the paper with a nice expression of generalized Somos' constants  $\sigma_b$  using values of the generalized Euler-constant function

$$\gamma(z) = \sum_{i=1}^{\infty} z^{i-1} \left( \frac{1}{i} - \log\left(\frac{i+1}{i}\right) \right),$$

where  $|z| \leq 1$ .

**Proposition 3.1** *For all integers  $b \geq 2$  we have*

$$\sigma_b = \frac{b}{b-1} e^{-\gamma(1/b)/b}.$$

**Proof.** We have

$$\gamma(1/b) = \sum_{i=1}^{\infty} (b^{-i+1}/i - b^{-i+1} \log(i+1) + b^{-i+1} \log(i))$$

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<sup>1</sup>They consider  $\sqrt[b-1]{\sigma_b}$ .

$$\begin{aligned}
&= b\left(\sum_{i=1}^{\infty} b^{-i}/i - \sum_{i=1}^{\infty} b^{-i} \log(i+1) + \sum_{i=1}^{\infty} b^{-i} \log(i)\right) \\
&= b(\log(b/(b-1)) - b\sigma_b/(b-1) + \sigma_b/(b-1)) = b\log(b/((b-1)\sigma_b))
\end{aligned}$$

using

$$\sum_{i=1}^{\infty} b^{-i}/i = \log(b/(b-1)) \text{ and } \log(\sigma_b) = \sum_{i=1}^{\infty} b^{-i}(b-1) \log(i).$$

Hence  $e^{\gamma(1/b)} = (b/((b-1)\sigma_b))^b$  and  $e^{\gamma(1/b)/b} = b/((b-1)\sigma_b)$  given  $\sigma_b = be^{-\gamma(1/b)/b}/(b-1)$ .  $\square$

Estimates of  $\gamma(1/b)$  can be found in [4].

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