Non-uniform expansions of real numbers

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Abstract

We introduce and study non-uniform expansions of real numbers, given by two non-integer bases.

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1 Introduction

Expansions of reals numbers in non-integer bases are studied since the pioneering works of Rényi in the end of the 1950s and Parry in the 1960s, see [11, 12, 13]. In these works especially the greedy algorithm that determines the digits of such expansions and the relationship of these expansions to symbolic dynamics is addressed. In the 1990s a group of Hungarian mathematics led by Paul Erdős revived this file of research, see [2, 4, 5]. Beside other results they proved that each $x \in (0, 1/(1-q))$ has a continuum of expansions of the form $\sum_{i=1}^{\infty} q^{-n_i}$ if $1 < q < G$, where G is the golden ratio. In the sequel Sidorov [14] used ergodic theoretical methods to prove that for all $q \in (1, 2)$ almost all $x \in (0, 1/(1-q))$ have such an expansion. Moreover, Glendinning and Sidorov [7] proved that there always exist (at least countably many) reals having a unique expansion if $q > G$. Nowadays especially dimensional theoretical aspects of expansions of reals numbers in non-integer bases are studied, see for instance [9, 10, 1].

In this paper we introduce non-uniform expansions of real numbers, which may be viewed as expansions with respect to two non-integer bases. As far as we know such expansions were not studied yet, although they constitute a natural generalisation. The rest of the paper is organized as follows:

In the next section we give two descriptions of non-uniform expansions of real numbers. In the following we introduce a greedy, a lazy and intermittent algorithms that give the digits of these expansions. In section four we prove a theorem on the existence of a continuum of non-uniform expansions of real numbers, which is similar to the results in the uniform case we mentioned above. In the last section we characterise real numbers which have a unique non-uniform expansion and prove a theorem on the cardinality of the set of such numbers.

2 The expansions

Let $\Sigma = \{0,1\}^{\mathbb{N}}$ be the set of sequences of zeros and ones. Equipped with the metric

$$d((s_i),(t_i)) = \sum_{i=1}^{\infty} |s_i - t_i| 2^{-i}$$

Σ is a compact, perfect and totally disconnected space. For $s = (s_i) \in \Sigma$ and $n \in \mathbb{N}$ let $0_n(s)$ be the number of zeros and $1_n(s)$ be the number of ones in the sequence $(s_1, \ldots, s_n)$. Fix $\beta_0, \beta_1 \in (1/2, 1)$ with $\beta_0 \geq \beta_1$ and let $I = I_{\beta_1} = [0, \beta_1/(1-\beta_1)]$.[1] We consider the map $\pi_{\beta_0,\beta_1} : \Sigma \to I$ given by

$$\pi_{\beta_0,\beta_1}(s) = \pi_{\beta_0,\beta_1}((s_i)) = \sum_{i=1}^{\infty} s_i \beta_0^{0_i(s)} \beta_1^{1_i(s)}.$$

Several times we will use another description of this map, which we now describe. Let $T_0, T_1 : I \to I$ be the contractions given by

$$T_0(x) = \beta_0 x \qquad \text{and} \qquad T_1(x) = \beta_1 x + \beta_1.$$

By induction we have

$$T_{s_1} \circ \cdots \circ T_{s_n}(x) = \beta_0^{0_n(s)} \beta_1^{1_n(s)} x + \sum_{i=1}^{n} s_i \beta_0^{0_i(s)} \beta_1^{1_i(s)}.$$

Hence for all $s \in \Sigma$ and all $x \in I$

$$\pi_{\beta_0,\beta_1}(s) = \pi_{\beta_0,\beta_1}((s_i)) = \lim_{n \to \infty} T_{s_1} \circ \cdots \circ T_{s_n}(x).$$

Definition 2.1 We call a sequence $s \in \Sigma$ with $\pi_{\beta_0,\beta_1}(s) = x$ a (β_0, β_1)-expansion of $x \in I$.

The following proposition guarantees the existence of (β_0, β_1)-expansions.

Proposition 2.1 *The map π_{β_0,β_1} is continuous and surjective.*

Proof. If $d((s_i),(t_i)) < 2^{-u}$ we have $s_i = t_i$ for $i = 1, \ldots, u$, which implies

$$|\pi_{\beta_0,\beta_1}(s_i) - \pi_{\beta_0,\beta_1}(t_i)| < \beta_0^u \beta_1/(1-\beta_1).$$

Hence π_{β_0,β_1} is continuous. Note that

$$T_0(I) \cup T_1(I) = [0, \beta_0\beta_1/(1-\beta_1)] \cup [\beta_1, \beta_1/(1-\beta_1)] = I$$

[1] In the literature the uniform case $\beta_0 = \beta_1$ has been studied. Usually the reciprocal of β_0 resp. β_1 is denoted by β.

since $\beta_0 + \beta_1 \geq 1$. This implies

$$\bigcup_{(s_i)\in\{0,1\}^n} T_{s_1} \circ \cdots \circ T_{s_n}(I) = I.$$

for every $n \geq 1$. Hence for each $x \in I$ there is sequence $s \in \Sigma$ such that

$$x \in T_{s_1} \circ \cdots \circ T_{s_n}(I),$$

By the compactness of Σ the map π_{β_0,β_1} is surjective $\square$

In the next section we describe an algorithm which determines one (β_0, β_1)-expansion of $x \in I$.

3 The greedy and the lazy algorithm

Using the notations of last section we define a map $G : I \to I$ by

$$G(x) = \begin{cases} T_0^{-1}(x), & x \notin T_1(I) \\ T_1^{-1}(x), & x \in T_1(I) \end{cases}$$

$$= \begin{cases} \beta_0^{-1}x, & x \in [0, \beta_1) \\ \beta_1^{-1}x - 1, & x \in [\beta_1, \beta_1/(1-\beta_1)], \end{cases},$$

see figure 1. For $x \in I$ we define the greedy expansion $g = g(x) = (g_i) \in \Sigma$ with respect to (β_0, β_1) by

$$g_i = \lfloor \beta_1^{-1} G^{i-1}(x) \rfloor,$$

where $\lfloor a \rfloor$ is the greatest integer not greater than a. We have

Proposition 3.1 *For all $x \in I$ the greedy expansion $g(x) \in \Sigma$ with respect to (β_0, β_1) is an (β_0, β_1)-expansion of x, that is $\pi_{\beta_0,\beta_1}(g(x)) = x$.*

Proof. If $g_i = 0$ we have $G^{i-1}(x) < \beta_1$, which implies $G^{i-1}(x) \notin T_1(I)$ and $G^{i-1}(x) \in T_{g_i}(I)$. If $g_i = 1$ we have $G^{i-1}(x) \geq \beta_1$, which implies $G^{i-1}(x) \in T_1(I)$ hence again $G^{i-1}(x) \in T_{g_i}(I)$. By the definition of G we conclude

$$x \in T_{g_1} \circ \cdots \circ T_{g_i}(I)$$

for all $i \in \mathbb{N}$, but this implies $\pi_{\beta_0,\beta_1}(g(x)) = x$. $\square$

To define the lazy expansion let $L : I \to I$ be given by

$$L(x) = \begin{cases} T_0^{-1}(x), & x \in T_0(I) \\ T_1^{-1}(x), & x \notin T_0(I) \end{cases}$$

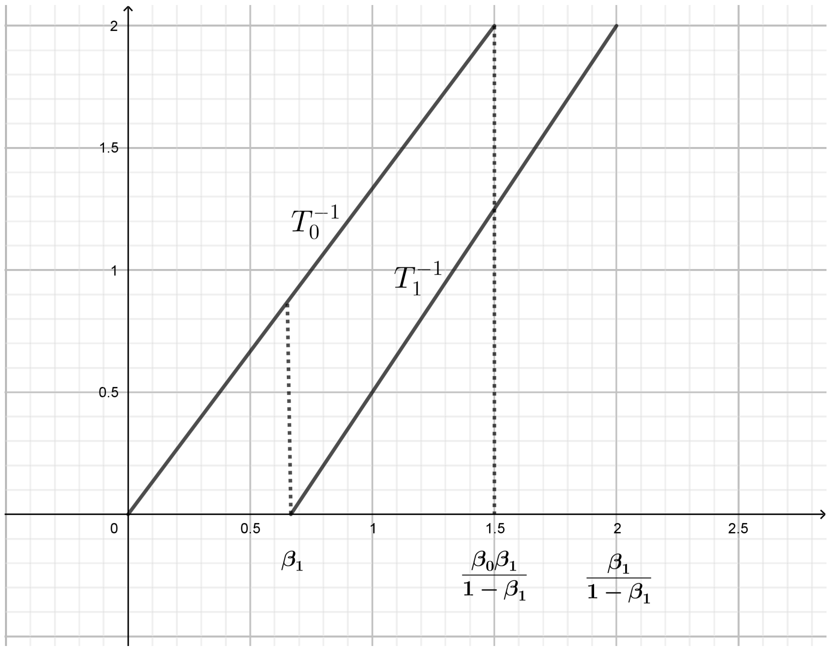

Figure 1: The maps T_0^{-1} and T_1^{-1} in the case $\beta_0 = 3/4, \beta_1 = 2/3$

$$= \begin{cases} \beta_0^{-1}x, & x \in [0, \beta_0\beta_1/(1-\beta_1)] \\ \beta_1^{-1}x - 1, & x \in (\beta_0\beta_1/(1-\beta_1), \beta_1/(1-\beta_1)], \end{cases},$$

see figure 1. For $x \in I$ the lazy expansion $l = l(x) = (l_i) \in \Sigma$ with respect to (β_1, β_2) is given by

$$l_i = \lceil (1-\beta_1)(\beta_0\beta_1)^{-1} L^{i-1}(x) \rceil - 1,$$

where $\lceil a \rceil$ is the smallest integer not smaller than a. We have

Proposition 3.2 *For all $x \in I$ the lazy expansion $l(x) \in \Sigma$ with respect to (β_1, β_2) is an (β_0, β_1)-expansion of x, that is $\pi_{\beta_0,\beta_1}(l(x)) = x$.*

Proof. If $l_i = 0$ we have $L^{i-1}(x) \le \beta_0\beta_1/(1-\beta_1)$, which implies $L^{i-1}(x) \in T_0(I)$ hence $L^{i-1}(x) \in T_{l_i}(I)$. If $l_i = 1$ we have $L^{i-1}(x) > \beta_0\beta_1/(1-\beta_1)$, which implies $L^{i-1}(x) \notin T_0(I)$ hence again $L^{i-1}(x) \in T_{l_i}(I)$. By the definition of L we conclude

$$x \in T_{l_1} \circ \cdots \circ T_{l_i}(I)$$

for all $i \in \mathbb{N}$, which implies $\pi_{\beta_0,\beta_1}(l(x)) = x$. $\square$

For $\alpha \in (\beta_1, \beta_0\beta_1/(1-\beta_1))$ we may also consider intermediate expansions $m(x) = (m_i)$ with respect to (β_1, β_2) given by

$$m_i = \lfloor \alpha^{-1} M_\alpha^{i-1}(x) \rfloor,$$

where

$$M_\alpha(x) = \begin{cases} \beta_0^{-1}x, & x \in [0,\alpha) \\ \beta_1^{-1}x - 1, & x \in [\alpha, \beta_1/(1-\beta_1)] \end{cases}.$$

Again these are (β_0, β_1)-expansions of $x \in I$.

4 A continuum of expansions

It is natural to ask how many (β_0, β_1)-expansions a real number in I has. It turns out that usually there is a continuum of such expansions:

Theorem 4.1 *Let $\beta_0, \beta_1 \in (1/2, 1)$ with $\beta_0 \geq \beta_1$. We have:*
(1) Almost all $x \in I$ have a continuum of (β_0, β_1)-expansions.
(2) If $\beta_1^2 + \beta_0 > 1$ all $x \in I \backslash \{0, \beta_1/(1-\beta_1)\}$ have a continuum of (β_0, β_1)-expansions.

Proof. We first prove (2). Let $J = (0, \beta_1/(1-\beta_1))$ and

$$\Lambda_0 = T_0(J) \cap T_1(J) = (\beta_1, \beta_0 \frac{\beta_1}{1-\beta_1}).$$

We recursively define $\Lambda_{n+1} = T_0(\Lambda_n) \cup \Lambda \cup T_1(\Lambda_n)$ and prove by induction:

$$\Lambda_n = (\beta_0^n \beta_1, (\beta_1^n(\beta_0 - 1) + 1)\frac{\beta_1}{1-\beta_1}).$$

We have

$$T_0(\Lambda_n) \cup \Lambda_n \cup T_0(\Lambda_n) = (\beta_0^{n+1}\beta_1, \beta_0(\beta_1^n(\beta_0 - 1) + 1)\frac{\beta_1}{1-\beta_1}) \cup (\beta_0^n \beta_1, (\beta_1^n(\beta_0 - 1) + 1)\frac{\beta_1}{1-\beta_1})$$

$$\cup (\beta_0^n \beta_1^2 + \beta_1, (\beta_1^{n+1}(\beta_0 - 1) + 1)\frac{\beta_1}{1-\beta_1}) = \Lambda_{n+1}.$$

In the last equation we use $\beta_1^2 + \beta_0 > 1$ and $\beta_0^2 + \beta_1 > 1$, which is true since $\beta_0 \geq \beta_1$. Note that $\bigcup_{n \geq 0} \Lambda_n = J$. Hence for every $x \in J$ there is a $k \geq 0$ and a sequence $(s_1, \ldots, s_k) \in \{0,1\}^k$ such that

$$x = T_{s_1} \circ \cdots \circ T_{s_k} \circ T_0(x_0) \text{ and } x = T_{s_1} \circ \cdots \circ T_{s_k} \circ T_1(x_1),$$

where $x_0, x_1 \in J$ and $x_0 \neq x_1$. Hence we obtain two expansions of x that differ in the $k+1$-digit. Applying the result to $x_0(x)$ and $x_1(x)$ we obtain four expansions of x. Here we use that $x_0(x)$ and $x_1(x)$ are not at the boundary of J. Repeating this procedure $\aleph_0$ times we see that there are $2^{\aleph_0}$ expansions of x.
Now we prove (1). Let $G : I \to I$ be the map associated with the greedy expansion from section 3. G is a piecewise linear expanding interval map and such maps are known to

have an ergodic measure, which is equivalent to the Lebesgue measure, see [3] and [8]. By Poincare recurrence theorem for almost all $x \in I$ there is a $k \geq 0$ such that $G^k(x) \in \Lambda_0$. Hence for almost all $x \in J$ there is a $k \geq 0$ and a sequence $(s_1, \ldots, s_k) \in \{0,1\}^k$ such that

$$x = T_{s_1} \circ \cdots \circ T_{s_k} \circ T_0(x_0) \text{ and } x = T_{s_1} \circ \cdots \circ T_{s_k} \circ T_1(x_1),$$

where $x_0, x_1 \in J$ and $x_0 \neq x_1$. For almost all x both numbers $x_1(x), x_2(x)$ have two different (β_0, β_1)-expansion hence almost all x have four different expansions. We use here that the intersection of two sets of full measure has full measure. Repeating this procedure $\aleph_0$ times we obtain $2^{\aleph_0}$ expansions for almost all $x \in I$, using the fact that a countable intersection of sets of full measure has full measure. □

Obviously the (β_0, β_1)-expansion of 0 and $\beta_1/(1-\beta_1)$ is unique. Our theorem leaves the question open if there are numbers x in the interior of I that have a unique (β_0, β_1)-expansion. We will address this question in the following section.

5 Unique expansions

We consider the shift map $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ given by $\sigma((s_k)) = (s_{k+1})$. Using this map we may characterise numbers which have a unique (β_0, β_1)-expansion as follows:

Proposition 5.1 *The (β_0, β_1)-expansion (s_i) of x is unique if and only if*

$$\pi_{\beta_0,\beta_1}(\sigma^k(s_i)) \in [0, \beta_1) \cup (\beta_0\beta_1/(1-\beta_1), \beta_1/(1-\beta_1)]$$

for all $k \geq 0$.

Proof. $\pi_{\beta_0,\beta_1}((s_i)) = \pi_{\beta_0,\beta_1}((t_i))$ with $(s_i) \neq (t_i)$ if and only if there exists a smallest $k \geq 0$ such that $s_{k+1} \neq t_{k+1}$ and $\pi_{\beta_0,\beta_1}(\sigma^k(s_i)) = \pi_{\beta_0,\beta_1}(\sigma^k(t_i))$. But this is equivalent to $\pi_{\beta_0,\beta_1}(\sigma^k(s_i)) \in T_0(I) \cap T_1(I) = [\beta_1, \beta_0\beta_1/(1-\beta_1)]$. The proposition follows by contraposition. □

Using this characterisation of points with unique expansion we are able to prove:

Theorem 5.1 *Let $\beta_0, \beta_1 \in (1/2, 1)$ and $\beta_0 \geq \beta_1$.*
(1) If $\beta_0(1+\beta_1) < 1$ there exist at least countable many $x \in I$, which have a unique (β_0, β_1)-expansion.
(2) If $\beta_0(1+2\beta_1-\beta_0\beta_1) < 1$ there are uncountable many $x \in I$, which have a unique (β_0, β_1)-expansion. Moreover the set of these x has positive Hausdorff dimension.

Proof. First we prove (1). Consider the periodic sequence $p = (010101\ldots)$. Since $\beta_0(1+\beta_1) < 1$ we have

$$\pi_{\beta_0,\beta_1}(p) = \beta_0\beta_1/(1-\beta_0\beta_1) < \beta_1.$$

Note that $\beta_0(1+\beta_1) < 1$ implies $\beta_1(1+\beta_0) < 1$ since $\beta_0 \geq \beta_1$. Hence we have $\beta_0 - \beta_0^2\beta_1 < 1 - \beta_1$ and thus

$$\pi_{\beta_0,\beta_1}(\sigma(p)) = \beta_1/(1-\beta_0\beta_1) = \beta_0\beta_1/(\beta_0 - \beta_0^2\beta_1) > \beta_0\beta_1/(1-\beta_1).$$

By proposition $x = \pi_{\beta_0,\beta_1}(p)$ has a unique (β_0, β_1)-expansion. Obviously the same is true for all x of the form $x = \pi_{\beta_0,\beta_1}((0\ldots0101010\ldots))$ and there exist countable many of such x.

Now we prove (2). Let $V = \{01, 10\}^{\mathbb{N}}$ and

$$U = \bigcup_{k=0}^{\infty} \sigma^k(V) = V \cup (\{0\} \times V) \cup (\{1\} \times V),$$

where $\times$ is the cartesian product. We prove that $\pi_{\beta_0,\beta_1}(U) \subseteq [0, \beta_1) \cup (\beta_0\beta_1/(1-\beta_1), \beta_1/(1-\beta_1)]$. The sequence $s \in U$ with $s_1 = 0$ that has the largest projection under π_{β_0,β_1} obviously is $s = (011010101\ldots)$. We have

$$\pi_{\beta_0,\beta_1}(s) = \beta_1 \frac{\beta_0 + \beta_0\beta_1 - \beta_0^2\beta_1}{1-\beta_0\beta_1} < \beta_1$$

by our assumption. The sequence $s \in U$ with $s_1 = 1$ that has the smallest projection under π_{β_0,β_1} obviously is $s = (1001010101\ldots)$. We have

$$\pi_{\beta_0,\beta_1}(s) = \beta_1 + \frac{(\beta_0\beta_1)^2}{1-\beta_0\beta_1} > \frac{\beta_0\beta_1}{1-\beta_1}.$$

The inequality here is equivalent to $\beta_1(1 + 2\beta_0 - \beta_0\beta_1) < 1$ which is true since we assume $\beta_0 \geq \beta_1$. It remains to show that the Hausdorff dimension of $A := \pi_{\beta_0,\beta_1}(V)$ is positive. Consider the maps

$$F(x) = T_0 \circ T_1(x) = \beta_0\beta_1 x + \beta_0\beta_1$$

and

$$H(x) = T_1 \circ T_0(x) = \beta_0\beta_1 x + \beta_1$$

and let $\Lambda = (\pi_{\beta_0,\beta_1}((01)^\infty), \pi_{\beta_0,\beta_1}((10)^\infty)$. We have $F(\Lambda) \subseteq \Lambda$ and $H(\Lambda) \subseteq \Lambda$ and $F(\Lambda) \cap H(\Lambda) = \emptyset$ by our assumptions on β_0 and β_1. In the language of fractal geometry this means that (F, H) induce an iterated function system fulfilling the open set condition, see [6]. The attractor of this iterated function system is A since $A = F(A) \cup H(A)$ and the classical formula for self-similar fractals gives

$$\dim_H A = \frac{-\log(2)}{\log(\beta_0\beta_1)} > 0.$$

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