Dimensionaltheoretical results for a family of generalized continued fractions

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Abstract

We find upper and lower estimates on the Hausdorff dimension of the set of real numbers, which have coefficients in a generalized continued fractions expansion that are bounded by a constant. As a consequence we prove a version of Jarnik's theorem; the set of real numbers with bounded coefficients in their generalized continued fractions representation has Hausdorff dimension one.

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1 Introduction

One highlight in metric theory of continued fractions is, that the set of real numbers with bounded coefficients in their continued fraction expansion has Lebesgue measure zero but Hausdorff dimension one. Khintchine [12] proved that the geometric mean of the coefficients in the continued fraction expansion converges to a constant for almost all real numbers. On the other hand Jarnik [10] found upper and lower and estimates on the Hausdorff dimension of the set of real numbers with continued fraction coefficients bounded by a constant and these estimates tend to one if we enlarge the bound. Over the years the approximation of the Hausdorff dimension of the set of real numbers with bounded continued fraction coefficients was addressed by several authors, see [7, 2, 3, 8, 9]. Today we know 54 digits of the Hausdorff dimension of the set of continued fractions with coefficients one and two, see [11].

In this work we consider generalized continued fractions associated with powers of integers $p \geq 2$. As far as we know these generalized continued fractions were for the first time considered by Chan [4, 5]. In [5] a Gauss-Kuzmin theorem on the distribution of coefficients in the case $b = 2$ is derived. In recent works Lascu [13, 14] proved a Gauss-Kuzmin theorem for all $b \geq 2$ and found an explicit expression for the Khintchine constant in this setting. These results seem to be rediscovered and generalized in an unpublished preprint, see [1].

We are interested in generalized continued fractions from a dimensional theoretical perspective. We find upper and lower estimates on the Hausdorff dimension of the set of real numbers which have bounded coefficients in their representation, see theorem 2.1 below. As a consequence of this result we get numerical approximations of the dimension for small bases b, which we list in table 1 below. Moreover we show that enlarging the bound on the coefficients the Hausdorff dimension tends to one. This means we obtain Jarnik's result for a family of generalized continued fractions, see theorem 2.2 and corollary 1 below.

2 Notations and Results

Let $b \geq 2$ be an integer. Almost all real numbers $x \in [0, 1)$ have unique representation in the form

$$x = [a_1, a_2, a_3, a_4, \ldots]_b := \cfrac{b^{-a_1}}{1 + \cfrac{(b-1)b^{-a_2}}{1 + \cfrac{(b-1)b^{-a_3}}{1 + \cdots}}},$$

with $a_i \in \mathbb{N}_0$, see [5]. The coefficients are given by the recursion

$$y_0 = x, \quad a_n = \lfloor \log_b(1/y_n) \rfloor, \quad y_{n+1} = \frac{1 - b^{a_n} y_n}{(b-1)b^{a_n} y_n}$$

and the generalized continued fraction is finite if and only if x is rational.

We are interested in the set of real numbers where the coefficients a_i in this representation are bounded by $n \geq 1$;

$$\mathfrak{F}_b(n) := \{[a_1, a_2, a_3, a_4 \ldots]_b \mid 0 \leq a_i \leq n\}.$$

It turns out that these sets have Lebesgue measure zero and we study their Hausdorff dimension. Let us recall that the d-dimensional Hausdorff measure of a set $B \subseteq \mathbb{R}$ is given by

$$H^d(B) = \lim_{\epsilon \to 0} \inf\{\sum_{i=1}^{\infty} (b_i - a_i)^d \mid B \subset \bigcup_{i=1}^{\infty} [a_i, b_i];\ \forall i \in \mathbb{N} : (b_i - a_i) \leq \epsilon\}$$

and the Hausdorff dimension of B is

$$\dim_H B = \inf\{d \mid H^d(B) = 0\} = \sup\{d \mid H^d(B) = \infty\}.$$

We recommend [6] or [16] as an introduction to dimension theory.

Our first main result provides upper and lower estimates on the Hausdorff dimension of the sets $\mathfrak{F}_b(n)$.

Theorem 2.1 *Fix integers $b \geq 2$ and $n \geq 1$. Let $U = U_b(n)$ be the positive solution of*

$$(b-1)b^n x^2 + ((2-b)b^n + b - 1)x - b^n = 0$$

and $L = L_b(n)$ be the positive solution of

$$(b-1)b^n x^2 + (b^{n+1} - b + 1)x - 1 = 0.$$

For $c \in [(b-1)/b, b-1]$ let $d = d(c)$ be the positive solution of

$$\sum_{i=0}^{n} \left(\frac{b-1}{b^i} \frac{(cU+1)(cL+1)}{(c/b^i + (b-1)U + 1)(c/b^i + (b-1)L + 1)} \right)^d = 1.$$

With these stipulations we have

$$\min\{d(c) \mid c \in [(b-1)/b, b-1]\} \leq \dim_H \mathfrak{F}_b(n) \leq \max\{d(c) \mid c \in [(b-1)/b, b-1]\}.$$

Using this theorem we get numerical approximations of the Hausdorff dimension of the sets $\mathfrak{F}_b(n)$ for small n and b. $\dim_H \mathfrak{F}_b(n)$ is within the open intervals given in the following table.

	b=2	b=3	b=4	b=5
n=1	(0.6070,0,6082)	(0.7369,0.7409)	(0.8040,0.8105)	(0.8445,0.8528)
n=2	(0.8289,0.8298)	(0.9232,0.9248)	(0.9568,0.9586)	(0.9741,0.9725))
n=3	(0.9219,0.9223)	(0.9762,0.9767)	(0.9899,0.9902)	(0.9948,0.9951)
n=4	(0.9632,0.9633)	(0.9924,0.9925)	(0.9976,09975)	(0.9989,0.9991)
n=5	(0.9822,0.9824)	(0.9975,0.9976)	(0.9993,0.9994)	(0.9997,0.9998)

Table 1: Upper and lower estimates on $\dim_H \mathfrak{F}_b(n)$

With some additional work we deduce from theorem 2.1 the following result on the Hausdorff dimension of $\mathfrak{F}_b(n)$ for all bases $b \geq 2$.

Theorem 2.2 *For all integers $b \geq 2$ the sequence $\dim_H \mathfrak{F}_b(n)$ converges to one from below.*

Now consider the sets of generalized continued fractions with bounded coefficients

$$\mathfrak{B}_b := \bigcup_{n=1}^{\infty} \mathfrak{F}_b(n).$$

Since Hausdorff dimension is countable stable the last theorem immediately implies:

Corollary 2.1 *For all $b \geq 2$ we have $\dim_H \mathfrak{B}_b = 1$.*

3 Preliminaries

In this section we collect basic results which will be used in the proof of theorem 2.1 and theorem 2.2. For integers $b \geq 2$ and $i \geq 0$ we consider the maps

$$T_i(x) = \frac{1}{b^i} \frac{1}{(1+(b-1)x)}$$

on $[0,1]$. Using these maps we obtain the following description of set of numbers $\mathfrak{F}_b(n)$ with bounded coefficients in their representation.

Proposition 3.1 *For integers $b \geq 2$ and $n \geq 1$ the set $\mathfrak{F}_b(n)$ is the invariant set for the iterated function system*

$$([L_b(n), U_b(n)], \{T_i \mid i = 0, \ldots, n\})$$

where $L_b(n)$ $U_b(n)$ are defined in theorem 2.1. This means $T_i([L_b(n), U_b(n)]) \subseteq [L_B(n), U_b(n)]$ and

$$\mathfrak{F}_b(n) = \bigcup_{i=0}^{n} T_i(\mathfrak{F}_b(n)).$$

Moreover this iterated function system fulfills the open set condition

$$T_i\left((L_b(n), U_b(n)\right) \cap T_j\left((L_b(n), U_b(n))\right) = \emptyset$$

for $i, j \in \{0, \ldots, n\}$ with $i \neq j$.

Proof. A simple calculation shows that $L_b(n)$ is given by the periodic continued fraction $[n, 0, n, 0, \ldots]_b$ and $U_b(n)$ is given by the periodic continued fraction $[0, n, 0, n, \ldots]_b$. We have

$$L_b(n) \leq [i, 0, n, 0, n, \ldots]_b = T_i(U_b(n)) < [i, n, 0, n, 0, \ldots]_b = T_i(L_b(n)) \leq U_b(n)$$

for all $i = 0, \ldots, n$. By continuity of T_i we get $T_i([L_b(n), U_b(n)]) \subseteq [L_B(n), U_b(n)]$. Moreover we have

$$\bigcup_{i=0}^{n} T_i(\mathfrak{F}_b(n)) = \bigcup_{i=0}^{n} \{[i, a_1, a_2, a_3, a_4 \ldots]_b \mid 0 \leq a_i \leq n\}$$
$$= \{[a_1, a_2, a_3, a_4 \ldots]_b \mid 0 \leq a_i \leq n\} = \mathfrak{F}_b(n).$$

If $x \in T_i\left((L_b(n), U_b(n)\right)$ and $j < i$ we have

$$x < T_i(L_b(n)) = [i, n, 0, n, 0, \ldots]_b \leq [j, 0, n, 0, n, \ldots]_b = T_j(U_b(n)) < T_j(L_b(n)),$$

hence $x \notin T_j\left((L_b(n), U_b(n)\right)$. This shows the last assertion in the proposition. $\square$

For a sequence $a = (a_1, \ldots, a_m) \in \{0, 1, \ldots, n\}^m$ we now consider the composition

$$T_a(x) := T_{a_m} \circ T_{a_{m-1}} \circ \cdots \circ T_{a_2} \circ T_{a_1}(x).$$

Using a recursion we get an explicit expression for these maps.

Proposition 3.2 *Fix integers $b \geq 2$ and $n \geq 1$. For all $a = (a_1, \dots, a_m) \in \{0, 1, \dots, n\}^m$ we have*

$$T_a(x) = \frac{(b-1)b^{a_{m-1}}\mu_{m-2}x + \mu_{m-1}}{(b-1)b^{a_m}\mu_{m-1}x + \mu_m},$$

where μ_j is given by the recursion

$$\mu_{j+1} = b^{a_{j+1}}(\mu_j + (b-1)\mu_{j-1})$$

with $\mu_{-1} = 0$ and $\mu_0 = 1$.

Proof. We prove the formula by induction. For $m = 1$ we have $\mu_1 = b^{a_1}$

$$T_{a_1}(x) = \frac{1}{b^{a_1}} \frac{1}{(1 + (b-1)x)} = \frac{(b-1)b^{a_0}\mu_{-1}x + \mu_0}{(b-1)b^{a_1}\mu_0 x + \mu_1}.$$

Assume that the formula folds for some m and let $a = (a_1, \dots, a_m)$ and $\tilde{a} = (a_1, \dots, a_m, a_{m+1})$. Using the assumption of the induction we have

$$T_{\tilde{a}}(x) = T_{a_{m+1}}(T_a(x)) = \frac{1}{b^{a_{m+1}}} \frac{1}{(1 + (b-1)T_a(x))}$$

$$= \frac{(b-1)b^{a_m}\mu_{m-1}x + \mu_m}{((b-1)b^{a_{m+1}})(b^{a_m}(\mu_{m-1} + (b-1)\mu_{m-2}))x + b^{a_{m+1}}(\mu_m + (b-1)\mu_{m-1})}$$

$$= \frac{(b-1)b^{a_m}\mu_{m-1}x + \mu_m}{(b-1)b^{a_{m+1}}\mu_m x + \mu_{m+1}}.$$

Here we use the recursive definition of μ_m and μ_{m+1} in the last equation. □

For further use we note:

Lemma 3.1 *Under the assumption of proposition 3.2 we have*

$$(b-1)b^{a_{m-1}}\mu_{m-2}\mu_m - (b-1)b^{a_m}\mu_{m-1}^2 = (1-b)^m b^{a_1+\dots+a_m}$$

Proof. For a fractional linear transformation $F(x) = (ax+b)/(cx+d)$ we consider the determinant of the corresponding matrix $|F| = ad - bc$. Since the composition of two fractional linear transformations acts like multiplication of the matrix of coefficients and the determinant is multiplicative we have $|F \circ G| = |F| \cdot |G|$. Since $|T_i| = (b-1)b^i$ the lemma now follows immediately from proposition 3.2. □

Using the last lemma and proposition 3.2 an obvious calculation gives:

Proposition 3.3 *Under the assumption of proposition 3.2 we have*

$$|T_a(x) - T_a(y)| = \frac{(1-b)^m b^{a_1+\dots+a_m}|x-y|}{((b-1)b^{a_m}\mu_{m-1}x + \mu_m)((b-1)b^{a_m}\mu_{m-1}y + \mu_m)}$$

In the proof of theorem 2.1 we use another lemma on the sequence μ_j.

Lemma 3.2 *If the sequence μ_j is given as in proposition 3.2 we have*

$$\frac{b-1}{b} \leq \frac{(b-1)b^{a_j}\mu_{j-1}}{\mu_j} \leq b-1.$$

Proof. By the definition of μ_j we have

$$\frac{(b-1)b^{a_j}\mu_{j-1}}{\mu_j} = \frac{(b-1)\mu_{j-1}}{\mu_{j-1}+(b-1)\mu_{j-2}}.$$

This gives the result, since μ_j is increasing. □

4 Proof of theorem 2.1

Fix integers $b \geq 2$ and $n \geq 1$ and define positive real numbers U and L as in theorem 2.1. By proposition 3.3 we have for all $m \geq 1$ and all sequences $a \in \{0,\dots,n\}^m$

$$S^\rho(a) := \sum_{i=0}^{n}\left(\frac{|T_a(T_i(U))-T_a(T_i(L))|}{|T_a(U)-T_a(L)|}\right)^\rho$$

$$= \sum_{i=0}^{n}\left(\frac{|T_i(U)-T_i(L)|}{|U-L|}\frac{((b-1)b^{a_m}\mu_{m-1}U+\mu_m)((b-1)b^{a_m}\mu_{m-1}L+\mu_m)}{((b-1)b^{a_m}\mu_{m-1}T_i(U)+\mu_m)((b-1)b^{a_m}\mu_{m-1}T_i(L)+\mu_m)}\right)^\rho$$

$$= \sum_{i=0}^{n}\left(\frac{b-1}{b^i}\frac{((b-1)b^{a_m}\mu_{m-1}U+\mu_m)((b-1)b^{a_m}\mu_{m-1}L+\mu_m)}{((b-1)b^{a_m}\mu_{m-1}\frac{1}{b^i}U+\mu_m)((b-1)b^{a_m}\mu_{m-1}\frac{1}{b^i}L+\mu_m)}\right)^\rho$$

$$= \sum_{i=0}^{n}\left(\frac{b-1}{b^i}\frac{(cU+1)(cL+1)}{(c/b^i+(b-1)U+1)(c/b^i+(b-1)L+1)}\right)^\rho$$

with

$$c = \frac{(b-1)b^{a_m}\mu_{m-1}}{\mu_m}.$$

By lemma 3.2 we have $c \in [(b-1)/b, b-1]$. By the definition of $d(c)$ in theorem 2.1 we hence have $S^{d(c)}(a) = 1$. Furthermore $d(c) \in [d_{\min}, d_{\max}]$ where

$$d_{\min} := \min\{d(c) \mid c \in [(b-1)/b, b-1]\} \text{ and } d_{\max} := \max\{d(c) \mid c \in [(b-1)/b, b-1]\}.$$

We thus obtain $S^{d_{\min}}(a) \leq 1$ and $S^{d_{\max}}(a) \geq 1$ for all $m \geq 1$ and all sequences $a \in \{0,\dots,n\}^m$. As a consequence there are constants $c, C > 0$ such that

$$\sum_{a\in\{0,\dots,n\}^m}(|T_a(U)-T_a(L)|)^{d_{min}} \leq C \quad\text{and}\quad \sum_{a\in\{0,\dots,n\}^m}(|T_a(U)-T_a(L)|)^{d_{max}} \geq c$$

for all $m \geq 1$. Theorem 2.1 now follows from theory of conformal iterated function systems fulfilling the open set condition, which may be applied by proposition 3.1, see [15]. We like to remark here that theorem 2.1 also follows from lemma 1 and lemma 2 in the work of Jarnik [10], which can obviously be generalized from classical continued fractions to generalized continued fractions.

5 Proof of theorem 2.2

For integers $b \geq 2$ and $n \geq 1$ we define $L_b(n)$ $U_b(n)$ as in theorem 2.1. First note that

$$\lim_{n\to\infty} L_b(n) = \lim_{n\to\infty} [n, 0, n, 0, \ldots]_b = 0$$

and

$$\lim_{n\to\infty} U_b(n) = \lim_{n\to\infty} [0, n, 0, n, \ldots]_b = 1.$$

Using these limits we have

$$\lim_{n\to\infty} \sum_{i=0}^{n} \left(\frac{b-1}{b^i} \frac{(cU_b(n)+1)(cL_b(n)+1)}{((b-1)U_b(n) + \frac{c}{b^i} + 1)((b-1)L_n(n) + \frac{c}{b^i} + 1)} \right)$$

$$= \sum_{i=0}^{\infty} \left(\frac{b-1}{b^i} \frac{(c+1)}{(\frac{c}{b^i} + 1)(\frac{c}{b^i} + b)} \right)$$

$$= \sum_{i=0}^{\infty} \left(\frac{b^{i+1}-1}{b^{i+1}+c} - \frac{b^i - 1}{b^i + c} \right) = \lim_{n\to\infty} \frac{b^n - 1}{b^n + c} = 1$$

for all $c \in \mathbb{R}$. Moreover if $c > 0$ all summands in the first expression are obviously positive hence the sequence approach one from below. Now choose $d(c) = d(c, n)$ as in theorem 2.1, that means

$$\sum_{i=0}^{n} \left(\frac{b-1}{b^i} \frac{(cU_b(n)+1)(cL_b(n)+1)}{((b-1)U_b(n) + \frac{c}{b^i} + 1)((b-1)L_n(n) + \frac{c}{b^i} + 1)} \right)^{d(c,n)} = 1.$$

We conclude that $d(c, n) < 1$ for all $c > 0$ and especially

$$\max\{d(c, n) \mid c \in [(b-1)/b, b-1]\} < 1$$

for all $n \geq 1$. Furthermore $\lim_{n\to\infty} d(c, n) = 1$ for all $c > 0$ and hence

$$\lim_{n\to\infty} \min\{d(c, n) \mid c \in [(b-1)/b, b-1]\} = 1.$$

Now theorem 2.2 follows from theorem 2.1.

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