Random walks on $\mathbb{Z}$ with exponentially increasing step length and Bernoulli convolutions

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Abstract

We establish a correspondence between the limit distribution and the asymptotic entropy of random walks on $\mathbb{Z}$, which have a sequence of step length that is exponentially increasing up to some error, and Bernoulli convolutions.

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1 Introduction

Since the pioneering work of Pòlya [14] discrete random walks are an important topic in probability theory. As an introductory example in textbooks we find the simple symmetric random walks on $\mathbb{Z}^d$ and especially the symmetric walk on $\mathbb{Z}$. In this article we consider symmetric random walks on $\mathbb{Z}$ with a sequence of step length that is exponentially increasing up to some error. We observe that these walks are related to infinite convolved Bernoulli measures, which are an imported subject of contemporary geometric measure theory. Since the work Jessen and Winter [8] and Erdös [3, 4] these measures are intensively studied. We refer to [13] for an overview about the first sixty years of research and to [16] for recent progress on these measures. We think that the correspondence we establish here is interesting because we get some information on the random walks we consider, which seems to be hard to obtain without using the correspondence.

We prove that the limit distribution of a random walk, which has exponentially increasing steps with a bounded series of errors scaled on an interval is given by an infinit convolved Bernoulli measure, see theorem 2.1 below. As a consequence we get a result on the asymptotic behaviours of the original walk. We discuss some examples associated with Pisot numbers and especially the Fibonacci walk. Moreover we show in this article that the asymptotic entropy of a random walk with exponentially increasing step length given by a lineare recursion is bounded by the entropy of a Bernoulli convolution, see theorem 4.1. If the step length are given by the sequence of n-bonacci numbers, the asymptotic entropy of the walk is in fact given by the entropy of a corresponding Bernoulli convolution, a quantity which is quite well known, see theorem 4.2.

The rest of the paper is organized as follows: In the next section we introduce some notations and formulate our result on the limit distribution of random walks with exponentially increasing step length. In section 3 we prove this result using a symbolic coding map. In section 4 we introduce the Shannon entropy and state our results on the entropy of radome walk. The last section contains the prove of this results using refitments of certain measurable partitions.

2 Limit distribution

Let (X_i) be the Bernoulli process consisting of i.i.d. random variables X_i with

$$P(X_i = 1) = P(X_i = -1) = 1/2.$$

Furthermore let μ be the corresponding Bernoulli measure on the sequence space $\{-1,1\}^{\mathbb{N}}$ equipped with the natural product metric. For $\beta \in (0,1)$ we define the infinit convolved Bernoulli measure μ_β on the interval $[-1,1]$ as the distribution of the random power series

$$X_\beta = \frac{1-\beta}{\beta}\sum_{i=1}^{\infty} X_i \beta^i,$$

that means $\mu_\beta(B) = \mathrm{Prob}(X_\beta \in B)$ for a Borel set B in $[-1,1]$. μ_β is usually called a Bernoulli convolution. We recommend again [13] and [16] as introductions to the theory of Bernoulli convolutions.

Let (a_i) be a sequence of positive integers and let $\hat{a}_n = a_1 + \cdots + a_n$. The random variables

$$\hat{X}_n = \sum_{i=1}^{n} a_i X_i$$

describe a random walk of n-steps on $\mathbb{Z}$ beginning at the origin where the i-th step of the walk is of length a_i with probability $1/2$ to the left and probability $1/2$ to the right. We recommend [12] as an introduction to the general theory of random walks. By $\hat{X}_n/\hat{a}_n$ we scale the random walk to interval $[-1,1]$. The scaled walk is supported by the set

$$\{\frac{1}{\hat{a}_n}\sum_{i=1}^{n} s_i a_i \mid s_i \in \{-1,1\}\}$$

of rationales in $[-1,1]$. It turns out that the limiting distribution of the scaled random walk is given by a Bernoulli convolution if the sequence a_i is exponentially increasing with a bounded series of errors. The precise result is given by the following theorem.

Theorem 2.1 *Let (a_i) be a sequence of positive integers such that there are constants $c > 0$ and $\beta \in (0,1)$ with*

$$\sum_{i=1}^{\infty} |a_i - c\beta^{-i}| < \infty.$$

For all Borel sets $B \subseteq [-1,1]$ we have

$$\mu\{(s_i) \in \{-1,1\}^{\mathbb{N}} \mid \lim_{n\to\infty} \frac{1}{\hat{a}_n} \sum_{i=1}^{n} s_i a_i \in B\} = \mu_\beta(B).$$

Let $F_{\hat{X}_\beta}$ be the cumulative distribution function of Bernoulli the convolution μ_β; that is $F_{\hat{X}_\beta}(x) = \mu_\beta([-1,x))$ Applying theorem 2.1 to the set $[-1,x)$ we obtain the following corollary on the asymptotic distribution of the unscaled random walk:

Corollary 2.1 *Under the assumption of theorem 2.1 we have*

$$\mu\{(s_i) \in \{-1,1\}^{\mathbb{N}} \mid \exists n_0 \forall n \geq n_0 \ : \ \sum_{i=1}^{n} s_i a_i \leq x\hat{a}_i\} = F_{\hat{X}_\beta}(x).$$

for all $x \in (-1,1)$.

Let us discuss some examples. For $a_i = a^{i-1}$, where a is an integer with $a \geq 2$, we have

$$\mu\{(s_i) \in \{-1,1\}^{\mathbb{N}} \mid \exists n_0 \forall n \geq n_0 \ : \ \sum_{i=1}^{n} s_i a^{i-1} \leq \frac{a^n-1}{a-1} x\} = F_{\hat{X}_{1/b}}(x).$$

For $b=2$ the measure $\mu_{1/2}$ ist just half of the Lebesgue measure and $F_{\hat{X}_{1/2}}(x) = 2x+1$. For $b \geq 3$ the measure $\mu_{1/b}$ is usually called a Cantor measure, it is supported by the Cantors set $\{\sum_{i=1}^{n} s_i a^{i-1} \mid s_i \in \{-1,1\}\}$. $F_{\hat{X}_{1/b}}$ is constant on open intervals outside this set, especially $F_{\hat{X}_{1/b}}(x) = 1/2$ for $x \in ((1-b)/b, (b-1)/b)$.
Now let $a_i = f_i$ be the Fibonacci sequence given by $f_{i+2} = f_{i+1} + f_i$ with $f_1 = f_2 = 1$. By the well known formula auf Faulhaber this sequence fulfills the condition of theorem 2.1 with $\beta = (\sqrt{5}-1)/2$ and we obtain

$$\mu\{(s_i) \in \{-1,1\}^{\mathbb{N}} \mid \exists n_0 \forall n \geq n_0 \ : \ \sum_{i=1}^{n} s_i f^i \leq (f_{n+2}-1)x\} = F_{\hat{X}_{(\sqrt{5}-1)/2}}(x).$$

The Bernoulli convolution $\mu_{\sqrt{5}-1)/2}$ has been intensively studied, see [1, 15, 7, 11]. It is a singular measure supported by $[-1,1]$ with Hausdorff dimension less than one. This example may be generalized in the following way: Let (a_i) be sequence of positive integers given by a lineare recurrence $a_{i+n} = c_n a_{i+n-1} + \cdots + c_1 a_i$ such that the characteristic polynomial $x^n = c_n x^{n-1} + \cdots + c_1 x^0$ has a unique dominating real root $\alpha = \beta^{-1} \in (1,2)$, which is a Pisot number. This means that all other roots of the polynomial have modulus less than one. In this case it is known that there are constance $a, b > 0$ and $\lambda \in (0,1)$ such that

$$|a_i - a\alpha^i| \leq b\lambda^i$$

for all $i \in \mathbb{N}$, moreover it is a classical result due to Pisot that the condition of our theorem even implies that β^{-1} is in a Pisot number. see [2]. The corresponding Bernoulli convolution μ_β is supported by $[-1,1]$ and known to be singular with Hausdorff dimension less than one, see [3, 5, 12]. In the next section the reader will find the proof of the theorem stated in this section.

3 Proof of theorem 2.1

To prove theorem 2.1 we define a coding $\mu_\beta : \{-1,1\}^{\mathbb{N}} \to [-1,1]$ by

$$\pi_\beta((s_i)) = \frac{1-\beta}{\beta} \sum_{i=1}^{\infty} s_i \beta^i.$$

We obviously have $\mu_\beta(B) = \mu(\pi_\beta^{-1}(B))$ for Borel sets $B \subseteq [-1,1]$. Theorem 2.1 is essentially based on the following proposition:

Proposition 3.1 *Under the assumption of theorem 2.1 we have*

$$\pi_\beta((s_i)) = \lim_{n\to\infty} \frac{1}{\hat{a}_n} \sum_{i=1}^{n} s_{n+1-i} a_i$$

for all $s_i \in \{-1,1\}^{\mathbb{N}}$.

Proof. Let $a_i = c\beta^{-i} + \epsilon_i$ we have

$$\hat{a}_n = \sum_{i=1}^{n} c\beta^{-i} + \epsilon_i = c\frac{\beta^{-(n+1)} - \beta^{-1}}{\beta^{-1} - 1} + \hat{\epsilon}_n = c\beta^{-n}\frac{1-\beta^n}{1-\beta} + \hat{\epsilon}_n,$$

where $\hat{\epsilon}_n = \sum_{i=1}^{n} \epsilon_i$. Using this we estimate the difference

$$D_n = |\frac{1-\beta}{\beta}\sum_{i=1}^{n} s_i\beta^i - \frac{1}{\hat{a}_n}\sum_{i=1}^{n} s_{n+1-i}a_i| = |\sum_{i=1}^{n} s_i \frac{1-\beta}{\beta}\beta^i - \sum_{i=1}^{n} s_i \frac{a_{n+1-i}}{\hat{a}_n}|$$

$$\le \sum_{i=1}^{n} |\frac{1-\beta}{\beta}\beta^i - \frac{a_{n+1-i}}{\hat{a}_n}| = \sum_{i=1}^{n} |\frac{1-\beta}{\beta}\beta^i - \frac{c\beta^{-i} + \epsilon_{n+1-i}}{c\beta^{-n}\frac{1-\beta^n}{1-\beta} + \hat{\epsilon}_n}|$$

$$= \sum_{i=1}^{n} |\frac{1-\beta}{\beta}\beta^i \left(1 - \frac{1}{1-\beta^n + \hat{\epsilon}_n\beta^n/c}\right) - \frac{(1-\beta)\beta^n\epsilon_{n+1-i}/c}{1-\beta^n + \hat{\epsilon}_n\beta^n/c}|$$

$$\le \sum_{i=1}^{n} |\frac{1-\beta}{\beta}\beta^i \left(1 - \frac{1}{1-\beta^n + \hat{\epsilon}_n\beta^n/c}\right)| + \sum_{i=1}^{n} |\frac{(1-\beta)\beta^n\epsilon_{n+1-i}/c}{1-\beta^n + \hat{\epsilon}_n\beta^n/c}|$$

$$\le n(1-\beta)\left|1 - \frac{1}{1-\beta^n + \hat{\epsilon}_n\beta^n/c}\right| + n(1-\beta)A\left|\frac{\beta^n/c}{1-\beta^n + \hat{\epsilon}_n\beta^n/c}\right|$$

$$\le n(1-\beta)\beta^n\left|\frac{1-\hat{\epsilon}_n/c}{1-\beta^n + \hat{\epsilon}_n\beta^n/c}\right| + n(1-\beta)A\beta^n\left|\frac{1/c}{1-\beta^n + \hat{\epsilon}_n\beta^n/c}\right|,$$

where $A = \sup\{|\epsilon_k| \mid k \ge 1\} < \infty$ by the assumption $\sum_{i=1}^{\infty} |\epsilon_i| < \infty$. Note that

$$|\hat{\epsilon}_n| = |\sum_{i=1}^{n} \epsilon_i| \le \sum_{i=1}^{n} |\epsilon_i| < \sum_{i=1}^{\infty} |\epsilon_i| = B < \infty$$

for all $n \geq 1$. Hence our estimate on D_n goes to zero for $n \to \infty$, which proves the proposition. $\square$

Proof of theorem 2.1. Let $B \subseteq [-1,1]$ be a Borel set and

$$U = \{(s_i) \in \{-1,1\}^{\mathbb{N}} \mid \lim_{n\to\infty} \frac{1}{\hat{a}_n} \sum_{i=1}^{n} s_i a_i \in B\}$$

$$V = \{(s_i) \in \{-1,1\}^{\mathbb{N}} \mid \lim_{n\to\infty} \frac{1}{\hat{a}_n} \sum_{i=1}^{n} s_{n+1-i} a_i \in B\}$$

By proposition 3.1 we have $V = \pi_\beta^{-1}(B)$, hence $\mu(V) = \mu_\beta(B)$. It remains to show that $\mu(U) = \mu(V)$. Consider cylinder sets of length n in $\{-1,1\}^{\mathbb{N}}$ given by

$$C(t_1,\ldots,t_n) = \{(s_i) \in \{-1,1\}^{\mathbb{N}} \mid s_i = t_i \text{ for } i = 1,\ldots,n\}.$$

A cylinder set $C(t_1,\ldots,t_n)$ is contained in U if and only if the cylinder set $C(t_n,\ldots,t_1)$ is contained in V. Therefore the number of cylinder sets of length n in U is equal to the number of these set V. The Bernoulli measure μ of a Borel set M in $\{-1,1\}^{\mathbb{N}}$ is given by

$$\mu(M) = \sup\{\sharp\{C \subseteq M | C \text{ a cylinder set of length } n\}/2^n \mid n \geq 1\}.$$

By continuity U and V are Borel measurable and we obtain $\mu(U) = \mu(V)$. $\square$

4 Entropy

Recall that the Shannon entropy of a random variabel X with a finite set of value in $\mathbb{R}$ is given by

$$H(X) = -\sum_{i \in X(\Omega)} P(X=i)\log(P(X=i)),$$

see 4.3 of [9] for instance. The entropy h_β of an infinite Bernoulli convolution μ_β for $\beta \in (0,1)$ is defined using the sequence of entropies of finite Bernoulli convolutions,

$$h_\beta = \lim_{n\to\infty} \frac{H(\sum_{i=1}^{n} X_i \beta^i)}{n}.$$

The limit exists since the sequence $H(\sum_{i=1}^{n} X_i \beta^i)$ is subadditiv, see [5]. We are interested here in the asymptotic entropy of a random walk on $\mathbb{Z}$ with exponential increasing step length. For such a sequences of step length given by a lineare recurrence the entropy of the walk is bounded by the entropy of the corresponding Bernoulli convolution, in the following sense:

Theorem 4.1 *Let (a_i) be an increasing sequence of positive integers fulfilling a linear recurrence $a_{i+n} = c_n a_{i+n-1} + \cdots + c_1 a_i$ with $c_i \in \{-1,0,1\}$ such that the characteristic* *polynomial* $x^n - c_n x^{n-1} - \cdots - c_2 x - c_1$ *is irreducible with dominating real root* $\alpha = \beta^{-1} \in (1,2)$. *Under these assumptions we have*

$$\limsup_{n\to\infty} \frac{H(\sum_{i=1}^n X_i a_i)}{n} \le h_\beta.$$

From the theory of Bernoulli convolutions we know that $h_\beta < \log\alpha$ if $\alpha = \beta^{-1} \in (1,2)$ is a Pisot number, this was originally proved by Grasia [5]. Hence we have

Corollary 4.1 *Under the assumption of theorem 4.1 and the assumption that* $\alpha \in (1,2)$ *is a Pisot number we have*

$$\limsup_{n\to\infty} \frac{H(\sum_{i=1}^n X_i a_i)}{n} < \log\alpha.$$

Without using the correspondence of random walks and Bernoulli convolutions this result seems to be hard to prove. In the special case of random walks whose step length is given by a n-bonacci sequences we are able to proof even a stronger result on the asymptotic entropy:

Theorem 4.2 *For* $n \ge 2$ *let* (a_i) *be the* n*-bonacci sequence given by the recurrence* $a_{i+n} = a_{i+n-1} + \cdots + a_i$ *for* $i \ge 0$ *and* $a_k = 2^k$ *for* $0 \le k < n$. *We have*

$$\lim_{n\to\infty} \frac{H(\sum_{i=1}^n X_i a_i)}{n} = h_{\beta_n},$$

where $\beta_n^{-1} \in (1,2)$ *is the domination real root of* $x^n - x^{n-1} - \cdots - x - 1 = 0$.

We note that the value of h_{β_n} is quite well known. In [1] and [15] formulae for h_{β_2} are given and we obtain

$$h_{\beta_2} / \log\beta_2^{-1} = 0.9957\ldots.$$

In [6] we find a formula for h_{β_n} for all $n \ge 2$. Especially we have

$$h_{\beta_3} / \log\beta_3^{-1} = 0.9804\ldots$$

$$h_{\beta_4} / \log\beta_4^{-1} = 0.9869\ldots$$

At the end of this section we like to ask if the last theorem may by generalized to other sequences associated with Pisot numbers. Our proof of the result in the next section is based on proposition which relies on the n-bonacci recurrence.

5 Proof of theorem 4.1 and 4.2

We use here partitions of the sequence $\{-1,1\}^{\mathbb{N}}$. For $\beta \in (0,1)$ and a sequence of positive integers (a_i) we define equivalence relations on $\{-1,1\}^{\mathbb{N}}$ by

$$\sum_{i=1}^k s_i \beta^i = \sum_{i=1}^k t_i \beta^i \qquad \text{and} \qquad \sum_{i=1}^k s_i a_i = \sum_{i=1}^k t_i a_i.$$

These relations induce partitions $\mathfrak{P}^k_\beta$ and $\mathfrak{P}^k_{(a_i)}$ of $\{-1,1\}^{\mathbb{N}}$ for $k \geq 1$. Note that we may express the entropy of Bernoulli convolution and random walks using these partitions

$$H(\mathfrak{P}^k_\beta) = -\sum_{P \in \mathfrak{P}^k_\beta} \mu(P) \log \mu(P) = H(\sum_{i=1}^{k} X_i \beta^i)$$

$$H(\mathfrak{P}^k_{(a_i)}) = -\sum_{P \in \mathfrak{P}^k_{(a_i)}} \mu(P) \log \mu(P) = H(\sum_{i=1}^{k} X_i a_i).$$

The proof of theorem 4.1 is based on proposition 6 of [10]. We reformulate the proposition according to our purposes in the following way

Proposition 5.1 *Let $\beta = \alpha^{-1}$ where $\alpha \in (1,2)$ be a root of an irreduzibel polynomial $x^n - c_n x^{n-1} - \cdots - c_2 x^1 - c_1$ with $c_i \in \{-1,0,1\}$. If we have*

$$\sum_{i=1}^{k} s_i \beta^i = \sum_{i=1}^{k} t_i \beta^i$$

with $s_i, t_i \in \{-1,1\}$ for $i = 1, \ldots, k$, then the sequence (t_i) may be obtained from the sequence (s_k) be a chain of additions of blocks of the form $\lambda(1, -c_n, \ldots, -c_2, -c_1)$ with $\lambda \in \mathbb{Z}$ at arbitrary positions.

Proof of theorem 2.1. Let us assume that

$$\sum_{i=1}^{k} s_i \beta^i = \sum_{i=1}^{k} t_i \beta^i,$$

where $s_i, t_i \in \{-1,1\}$. Recall that the sequence (a_i) fulfils $a_{i+n} - c_n a_{i+n-1} - \cdots - c_1 a_i = 0$. Hence proposition 4.1. implies

$$\sum_{i=1}^{k} s_{k+1-i} a_i = \sum_{i=1}^{k} t_{k+i-1} a_i.$$

We have thus shown that the partition $\mathfrak{P}^k_\beta$ is finer than the $\mathfrak{P}^k_{(a_i)}$. By well known properties of the Shannon entropy we thus have $H(\mathfrak{P}^k_\beta) \geq H(\mathfrak{P}^k_{(a_i)})$, see 4.3 of [9]. Dividing the inequality by k and considering the limit $k \to \infty$ proves the theorem. □

To prove theorem 2.2 we use an analogon of lemma 5.1 for n-bonacci sequences.

Proposition 5.2 *For $n \geq 2$ let (a_i) be the n-bonacci sequence given by the recurrence $a_{i+n} = a_{i+n-1} + \cdots + a_i$ for $i \geq 0$ and $a_k = 2^k$ for $0 \leq k < n$. If we have*

$$\sum_{i=1}^{k} s_i a_i = \sum_{i=1}^{k} t_i a_i$$

with $s_i, t_i \in \{-1, 1\}$ for $i = 1, \ldots, k$ than the sequence (t_i) may be obtained from the sequence (s_k) be a chain of substitutions of blocks $(-1, \ldots, -1, 1)$ *with blocks* $(+1, \ldots, +1, -1)$ *and substitutions vice versa.*

Proof. First note for that sums are equal for $k \leq n$ if and only if $s_i = t_i$ for $i = 1, \ldots, n$, since we have $a_k = 2^k$ for $0 \leq k < n$. For $k = n + 1$ the conclusion is obviously true since a_i is a n-bonacci sequence. We now prove the result using induction. Assume that the proposition is true for all $\bar{k} \leq k$ and assume that

$$\sum_{i=1}^{k+1} s_i a_i = \sum_{i=1}^{k+1} t_i a_i.$$

If $s_{k+1} = t_{k+1}$ the conclusion immediately follows from the induction hypothesis. Without loss of generality we may now assume $s_{k+1} = 1$ and $t_{k+1} = -1$. We show by contradiction that at least one of the sequences (s_k) and (t_k) ends with a block that may substituted. The result follows by the induction hypothesis if this is true. Let us assume that the sequence (s_k) does not end with $(-1, \ldots, -1, 1)$ and (t_k) does not end with $(1, \ldots, 1, -1)$. Under this assumption we have

$$\sum_{i=k-n}^{k+1} s_i a_i - \sum_{i=k-n}^{k+1} t_i a_i = \sum_{i=k-n}^{k} \left((s_i + 1) - (t_i - 1)\right) a_i \geq 4 a_{k-n}$$

$$> 2 \sum_{i=1}^{k-n-1} a_i \geq \sum_{i=1}^{k-n-1} t_i a_i - \sum_{i=1}^{k-n-1} s_i a_i,$$

which is a contradiction to the assumption. □

Proof of theorem 2.2. Let (a_i) be a n-bonacci sequence and assume that

$$\sum_{i=1}^{k} s_i a_i = \sum_{i=1}^{k} t_i a_i,$$

where $s_i, t_i \in \{-1, 1\}$. Recall that β_n^{-1} is the root of $x^n - x^{n-1} - \cdots - x - 1 = 0$. Proposition 5.2 gives

$$\sum_{i=1}^{k} s_{k+1-i} \beta_n^i = \sum_{i=1}^{k} t_{k+1-i} \beta_n^i.$$

This shows that the partition $\mathfrak{P}^k_{(a_i)}$ is finer than the partition $\mathfrak{P}^k_{\beta_n}$, which implies $H(\mathfrak{P}^k_{\beta_n}) \leq H(\mathfrak{P}^k_{(a_i)})$, see again 4.3 of [9]. Together with theorem 2.1 this estimate gives the result.□

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