Return of Fibonacci random walks

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Abstract

We determine the probability of return for random walks on $\mathbb{Z}$ whose increment is given by the Fibonacci sequence. It addition we calculate the Hausdorff dimension of the set of these walks that return an infinite number of times.

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1 Introduction

Let X_i be a sequence of independent, identically distributed random variables with $P(X_1 = \pm 1) = 1/2$. The simple symmetric random walk of length n on the integers is given by the random variables

$$\hat{X}_n = \sum_{i=1}^{n} X_i.$$

It is a classical result due to Polya [9] that this random walk is recurrent, it returns to the origin in a finite number of steps with probability one. The expected number of returns to the origin for the n-step walk growth as $\sqrt{n}$, see [3] for the exact result. Furthermore it is know the the expected distance to the origin growth with the same rate. We also like to mention a result that is not well known. The walks with prescribe frequencies of return to the origin are fractals in the metric space of ± 1 sequences; Mae and Wen [5] were able to determine the Hausdorf dimension of this sets.
In this paper we consider Fibonacci random walks given by

$$\hat{F}_n = \sum_{i=1}^{n} f_i X_i$$

where $f_1 = 1$, $f_2 = 2$ and $f_{i+1} = f_i + f_{i-1}$.[1] Viswanath [10] gave a computer assisted proof that the expected distance to the origin of this walk growth exponentially and estimates the base by 1.13198824. An elementary proof of exponential growth is given by Markover and Mcgowan [7]; they showed that the base is in $[1.12095, 1.23375]$. Both articles rely on a convergence theorem for random matrices given by Furstenberg [4].

[1] To avoid a trivial return at the beginning of the walk, we do not choose $f_1 = 1$ and $f_2 = 1$. We remark that the walk may be shown to be transient in this case as well.

As far is we know the return to the origin of Fibonacci random walks has not been studied. This will be done here. In contrast to the simple random walk the Fibonacci walk is transient. We are able to calculate the probability of i-returns to the origin for any $i \geq 0$, see theorem 2.1 below. The expected number of returns turns out to be 1/3. Moreover we consider the sets of all walks that return to the origin an infinite number of times. The set is a fractal in the metric space of ± 1 sequences with Hausdorff dimension 1/3, see theorem 3.1 below.

2 Finite number of returns

For $i \in \mathbb{N}_0$ we consider the event R_i that the Fibonacci random walk $\hat{F}_n$ defined in the last section reaches the origin exactly i times, that is

$$R_i = \{w \in \{-1,1\}^{\mathbb{N}} \mid \sharp\{n \geq 1 \mid \hat{F}_n(w) = 0\} = i\}$$

In section we will show:

Theorem 2.1 *The probability of R_i is $3/4^{i+1}$ for $i \in \mathbb{N}_0$.*

As a corollary we obtain

Corollary 2.1 *The expectation E of the number of returns of $\hat{F}_n$ to the origin is* $1/3$.

Proof. Just note that

$$E = \sum_{i=0}^{\infty} i\mathbb{P}(R_i) = \sum_{i=0}^{\infty} \frac{3i}{4^{i+1}} = 1/3.$$

□

The proof of theorem 2.1 is essentially based on the following property of the Fibonacci sequence, which seems not to be recognised before.

Proposition 2.1 *Let $(w_k) \in \{-1,1\}^{\mathbb{N}}$ and let f_k be the Fibonacci sequence $f_{k+1} = f_k + f_{k-1}$ with $f_1 = 1$ and $f_2 = 2$. We have*

$$\sum_{k=1}^{n} w_k f_k = 0$$

if and only if $n = 3m$ for some $m \geq 0$ and

$$(w_1, \dots, w_n) \in \{+1+1-1, -1-1+1\}^m$$

Proof. The "if" part of the theorem is obvious since $f_{i+1} - f_i - f_{i-1} = 0$ and $-f_{i+1} + f_i + f_{i-1} = 0$. To see the "only if" part we show that

$$\sum_{k=1}^{n} w_k f_k = 0$$

implies $(w_{n-2}w_{n-1}w_n) = (+1+1-1)$ or $(w_{n-2}w_{n-1}w_n) = (-1-1+1)$ for all $n \geq 3$. Since $w_1 f_1 + w_2 f_2 \neq 0$ the result follows by induction.
Assume that $(w_{n-2}w_{n-1}w_n) \neq (+1+1-1)$ and $(w_{n-2}w_{n-1}w_n) \neq (-1-1+1)$. This implies

$$|w_{n-2}f_{n-2} + w_{n-1}f_{n-1} + w_n f_n| = |(w_{n-2} + w_n)f_{n-2} + (w_{n-1} + w_n)f_{n-1}| \geq 2f_{n-2}.$$

On the other hand

$$|\sum_{k=1}^{n-3} w_k f_k| \leq \sum_{k=1}^{n-3} f_k = f_{n-1} - 2.$$

We have $2f_{n-2} \geq f_{n-1} - 2$ hence

$$\sum_{k=1}^{n} w_k f_k \neq 0.$$

Now the contraposition gives the implication. □

Proof of theorem 2.1. From the last proposition we infer that the random walk $\hat{F}_n$ reaches the origin exactly i-times if and only if $\hat{F}_{3n} = 0$ and $\hat{F}_{3n+3} \neq 0$. Moreover we have $\mathbb{P}(\hat{F}_{3n}) = (1/4)^n$ and $\mathbb{P}(\hat{F}_{3n+3} \neq 0 \mid \hat{F}_{3n} = 0) = 3/4$. This gives the result. □

3 Infinite number of returns

We consider the event N that the Fibonacci random walk $\hat{F}_n$ defined in the first section reaches the origin an infinite number of times, that is

$$N = \{w \in \{-1,1\}^{\mathbb{N}} \mid \sharp\{n \geq 1 \mid \hat{F}_n(w) = 0\} = \infty\}.$$

From the last section we know the the probability of this event is zero, nevertheless we may ask for the "size" of the set these random walks as subsets of the sequences space $\{-1,1\}^{\mathbb{N}}$. With the natural metric

$$d((w_k),(v_k)) = \sum_{k=1}^{\infty} |w_k - v_k| 2^{-k}$$

$\{-1,1\}^{\mathbb{N}}$ is a compact metric space. Hence we have the notation of the d-Hausdorff measure of N,

$$H^d(N) := \lim_{\epsilon \to 0} \inf\{\sum_{i=1}^{\infty} |C_i|^d \mid N \subseteq \bigcup_{i=1}^{\infty} C_i \ , \ |C_i| \leq \epsilon\},$$

and of the notion of Hausdorff dimension of N,

$$\dim_H N = \inf\{d \geq 0 \mid H^d(N) = 0\} = \sup\{d \geq 0 \mid H^d(N) = \infty\}.$$

We refer to [1] and [8] for an introduction to dimension theory. Here we obtain.

Theorem 3.1 *We have*

$$\dim_H N = 1/3.$$

The proof of this theorem is based on the following lemma, which follows directly from proposition 2.1.

Lemma 3.1 *We have*

$$N = \{+1+1-1, -1-1+1\}^{\mathbb{N}}.$$

Proof of theorem 3.1 Consider the maps

$$T_1((w_k)) = (+1+1-1w_1w_2,\dots) \quad T_2((w_k)) = (-1-1+1w_1w_2,\dots).$$

These are contracting similarities on the space $(\{-1,1\}^{\mathbb{N}}, d)$ with contraction rate 1/8. It follows from [2] that there is a unquiet compact self-similar set $S \subseteq \{-1,1\}^{\mathbb{N}}$ with

$$S = T_1(S) \cup T_2(S)$$

and by lemma 3.1 we habe $S = N$. Furthermore we have $T_1(N) \cap T_2(N) = \emptyset$. Hence N is a self-similar set fulfilling the open set condition in compact metric space. It is well known in dimension theory that the dimension of this set is given by

$$\dim_H N = \frac{\log(2)}{\log(8)} = 1/3.$$

In 9.1 of [1] the reader will find a proof of the dimension formula for self-similar sets in the euclidian space, which may be generalized to compact metric spaces. The dimension formula, also follows from theorem 3.15 of [6]. □

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