Singular Bernoulli convolutions for non-Pisot numbers

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Abstract

We show that biased Bernoulli convolutions get singular for non-Pisot numbers in the domain where these measures are generically absolutely continuous. In addition we compute families of such exceptions.

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1 Introduction and Results

For \( p \in (0, 1) \) and \( \beta \in (0, 1) \) we consider the infinite convolved Bernoulli measure (short: Bernoulli convolution)

\[
\mu_{\beta,p} = \ast_{n=0}^{\infty} (p \delta_{\beta n} + (1-p) \delta_{-\beta n}),
\]

where \( \delta \) is the Dirac measure. This is a law of pure type; the measure is either absolutely continues or singular with respect to the Lebesgue measure [8] and in the case of absolute continuity even equivalent to the Lebesgue measure [10]. It is folklore that the Bernoulli convolutions are singular for \( \beta < p \frac{1}{1-p} \) and it is known that the set of exceptions in the domain of absolutely continuity is small in the following sense:

**Theorem 1.1** There exists a set \( E \subseteq (1/2, 1) \) of Hausdorff dimension zero such that \( \mu_{\beta,p} \) is absolutely continuous for all \( p \in (0, 1) \) and all \( \beta \in (1/2, 1) \setminus E \) with \( \beta \geq p \frac{1}{1-p} \).

This result has quite a long history: Erdős [4] proved absolute continuity of \( \mu_{\beta,0.5} \) for almost all \( \beta \) in some interval \( (1-\delta, 1) \) and Solomyak [14] obtained absolute continuity for almost all \( \beta \in (0, 5, 1) \). Peres and Solmyak simplified the proof of this result using transversality techniques and proved absolute continuity of \( \mu_{\beta,p} \) for all \( p \in [1/3, 2/3] \) and almost all \( \beta \in (p \frac{1}{1-p} - p, 1) \), see [11, 12]. Than there is a recent breakthrough of Hochman [7] who used inverse theorems for entropy to prove \( \dim_H \mu_{\beta,p} = 1 \) for \( \beta \in (p \frac{1}{1-p} - p, 1) \) outside a set Hausdorff dimension zero. Shmerkin [13] used the result of Hochman to prove the theorem stated above.

The challenging question that remains open is to characterize the exceptional values \( \beta \) for which \( \mu_{\beta,p} \) is singular. As far as we know the only known result so far is due to Erdős [4]. Recall here that a Pisot number is an algebraic integer with all its conjugates inside the unite, see [2].
Theorem 1.2 If $\beta \in (0.5, 1)$ is the reciprocal of a Pisot number $\mu_{\beta,p}$ is singular for all $p \in (0, 1)$

In fact Erdős proved the theorem for $p = 0.5$ but the proof can by generalized easily. In this paper we will demonstrate that it is quite easy to get the singularity of $\mu_{\beta,p}$ for algebraic numbered $\beta \in (0.5, 1)$ that are not Pisot numbers and some $p$ with $p^p(1 - p)^{1-p} < \beta$.

Theorem 1.3 For all algebraic integers $\beta \in (0.5, 1)$ which full fill an algebraic equation with coefficients in $\{-1, 0, 1\}$ there is an $p = p(\beta) \in (0, 1/2]$ with $\beta > p^p(1 - p)^{(1-p)}$ such that $\mu_{\beta,p}$ is singular for all $p < p$.

Examples of algebraic integers for which our theorem applies are the reciprocals of $\sigma_n \in (1, 2)$, given by the dominating root of

$$x^n - x^{n-1} - \cdots - x + 1 = 0, \quad n \geq 4,$$

and the reciprocal of $\nu_n \in (1, 2)$, given by the dominating root of

$$x^n - x^{n-1} - \cdots - x^3 - 1 = 0, \quad n \geq 4.$$

Note that the numbers $\sigma_n$ are Salem numbers (the conjugates have absolute value no greater than 1, and at least one of them has absolute value exactly 1) and the numbers $\nu_n$ are neither Pisot nor Salem numbers for $n \geq 14$, see [2]. The mayor challenge that remains is to determine $p(\beta)$ if $\beta^{-1}$ is not a Pisot number. Our techniques only give lower bounds on $p(\beta)$, which we compute in the last section for reciprocals of algebraic integers of degree four. The next section is devoted to the proof of Theorem 3.1.

2 Proof of Theorem 4.1

For $\beta \in (0, 1)$ we define a sequence of relations $\sim_n$ on $\{-1, 1\}^N$ by

$$(s_i) \sim_n (t_i) :\leftrightarrow \sum_{i=1}^{n} s_i \beta^i = \sum_{i=1}^{n} t_i \beta^i.$$ 

Obviously this are equivalence relations. Let $P_n$ be the partition of $\{-1, 1\}^N$ induced by $\sim_n$ and let $b^p$ be the Bernoulli measure on $\{-1, 1\}^N$ for $p \in (0, 1)$. Define the entropy of the Partition $P_n$ with respect to the Bernoulli measure by

$$H_p(P_n) = -\sum_{P \in P_n} b^p(P) \log(b^p(P)).$$

Observe that the partition $P_n \vee \sigma^{-n} P_m$ is finer than $P_{n+m}$. Hence by well known properties of the partition entropy, $H_p(P_n)$ is a sub-additive sequence, see [15]. Hence the Garsia entropy

$$G_\beta(p) = \lim_{n \to \infty} \frac{H_p(P_n)}{n} = \inf \left\{ \frac{H_p(P_n)}{n} | n \geq 1 \right\},$$

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exists; compare with [5] and [6]. The following lemma is essential for our proof of theorem 1.3.

**Lemma 2.1** For all algebraic integers \( \beta \in (0.5, 1) \) which full fill an algebraic equation with coefficients in \( \{ -1, 0, 1 \} \) we have

\[
G_\beta(p) \leq U_\beta(p) < -p \log(p) - (1 - p) \log(1 - p)
\]

where \( U_\beta : (0, 1) \to \mathbb{R} \) is a continuous function.

**Proof.** By the assumption the partition \( C_n \) of \( \{ -1, 1 \}^N \) into cylinder sets is finer than the partition \( P_n \) if \( n \) is large enough. Hence

\[
H_p(P_n) < H_p(C_n) = -np \log(p) - n(1 - p) \log(1 - p).
\]

\( U_\beta(p) := H_p(P_n)/n \) is the continuous function we are searching for. \( \square \)

Beside this lemma we will use proposition 3 of [9]:

**Proposition 2.1** For all \( p \in (0, 1) \) and \( \beta \in (0, 1) \)

\[
\dim_H \mu_{\beta, p} \leq \frac{G_\beta(p)}{\log \beta^{-1}},
\]

where \( \dim_H \mu_{\beta, p} \) denotes the Hausdorff dimension of the measure.

**Proof of theorem 3.1** Fix \( \beta \) and choose \( U_\beta \) from lemma 2.1. Let

\[
p = \sup\{ \tilde{p} \in (0, 1/2] \mid U_\beta(p) < \log \beta^{-1} \ \forall p \in (0, \tilde{p}] \}.
\]

If \( p < p \) we obtain by lemma 2.1 and proposition 3.1

\[
\dim_H \mu_{\beta, p} \leq \frac{G_\beta(p)}{\log \beta^{-1}} \leq \frac{U_\beta(p)}{\log \beta^{-1}} < 1,
\]

hence \( \dim_H \mu_{\beta, p} \) is singular. If \( \beta \leq p^p(1 - p)^{1-p} \) for some \( p \), we have

\[
U_\beta(p) < -p \log(p) - (1 - p) \log(1 - p) \leq \log \beta^{-1}.
\]

By continuity of \( U_\beta(p) \) we get \( \beta > p^p(1 - p)^{(1-p)} \).

\[\square\]

### 3 Computational results

Computations show that there are six polynomials of degree four with coefficients in \( \{ -1, 0, 1 \} \) (and leading coefficient one) which have roots in \( (1, 2) \), that are not Pisot numbers. These are

\[
x^4 - x^3 - x^2 - x + 1, \quad x^4 - x^3 - x^2 + x - 1, \quad x^4 - x^3 + x^2 - x - 1,
\]
\[ x^4 + x^3 - x^2 - x - 1, \quad x^4 - x^2 - 1, \quad x^4 - x - 1 \]

We denote these roots by \( \alpha_i \) for \( i = 1, \ldots, 6 \). \( \alpha_1 \) is a Salem number and the other numbers are neither Pisot nor Salem numbers. We have computed the partitions \( P_{14} = P_{14}(\beta_i) \) defined in the last section with respect to \( \beta_i = \alpha_i^{-1} \) explicitly using Mathematica. Furthermore we computed the Bernoulli measures of the partition elements and thus obtain a lengthy but explicit expression for the entropy \( h_i(p) := H_p(P_{14}(\beta_i)) \). By proposition 2.1 and the definition of the Garcia entropy in the last section we have

\[
\dim_H \mu_{\beta_i, p} \leq h_i(p)/(14 \log(\alpha_i)).
\]

If we set

\[
p_i := \min\{p \geq 0 \mid h_i(p)/(14 \log(\alpha_i)) \geq 1\}
\]

the Bernoulli convolution \( \mu_{\beta_i, p} \) is singular with \( \dim_H \mu_{\beta_i, p} < 1 \) for all \( p \in [0, p_i) \). Using the explicit expression for \( h_i(p) \) we get the following approximations of \( p_i \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_i )</td>
<td>0.5806</td>
<td>0.6609</td>
<td>0.7748</td>
<td>0.8483</td>
<td>0.7861</td>
<td>0.8181</td>
</tr>
<tr>
<td>( p_i )</td>
<td>0.2492</td>
<td>0.1495</td>
<td>0.0708</td>
<td>0.0389</td>
<td>0.0696</td>
<td>0.0539</td>
</tr>
<tr>
<td>( p_i^p(1 - p_i)^{(1-p_i)} )</td>
<td>0.5703</td>
<td>0.6558</td>
<td>0.7743</td>
<td>0.8482</td>
<td>0.7767</td>
<td>0.8107</td>
</tr>
</tbody>
</table>

Note that we have

\[
\beta_i > p_i^p(1 - p_i)^{(1-p_i)}
\]

for all \( i = 1, \ldots, 6 \). Hence the algebraic integers \( \beta_i \) for \( i = 1, \ldots, 6 \) are exceptions in the domain of generic absolute continuity given by theorem 1.1.

At the end of this paper we like to remark that in a very resent preprint [1] the authors obtain full dimension of the unbiased Bernoulli convolutions \( \mu_{\beta_i, 0.5} \) for \( i = 2, \ldots, 6 \) using numerical computations. With their techniques we may also proof that \( \dim_H \mu_{\beta_i, p} = 1 \) for all \( p \) in some interval around 0.5 and \( i = 2, \ldots, 6 \). For the Salem number \( \alpha_1 = \beta_1^{-1} \) their approach does not work.

References


