Singular Bernoulli convolutions for non-Pisot numbers

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Abstract

We show that biased Bernoulli convolutions get singular for non-Pisot numbers in the domain where these measures are generically absolutely continuous. .

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1 Introduction and Results

For $p \in (0,1)$ and $\beta \in (0,1)$ we consider the infinite convolved Bernoulli measure (short: Bernoulli convolution)

$$\mu_{\beta,p} = *_{n=0}^{\infty}(p\delta_{\beta^n} + (1-p)\delta_{-\beta^n}),$$

where δ is the Dirac measure. This a law of pure type; the measure is either absolutely continues or singular with respect to the Lebesgue measure [8] and in the case of absolute continuity even equivalent to the Lebesgue measure [10]. It is folklore that the Bernoulli convolutions are singular for $\beta < p^p(1-p)^{1-p}$ and it is known the set of exceptions in the domain of absolutely continuity is small in the following sense:

Theorem 1.1 *There exists a set $E \subseteq (1/2,1)$ of Hausdorff dimension zero such that such that $\mu_{\beta,p}$ is absolutely continuous for all $p \in (0,1)$ and all $\beta \in (1/2,1)\backslash E$ with $\beta \geq p^p(1-p)^{1-p}$.*

This result has quite a long history: Erdös [3] proved absolute continuity of $\mu_{\beta,0.5}$ for almost all β in some interval $(1-\delta,1)$ and Solomyak [14] obtaiend absolute continuity for almost all $\beta \in (0,5,1)$. Peres and Solmyak simplified the proof of this result using transversality techniques and proved absolute continuity of $\mu_{\beta,p}$ for all $p \in [1/3,2/3]$ and almost all $\beta \in (p^p(1-p)^{1-p},1)$, see [11, 12]. Than there is a very recent breakthrough of Hochman [7] who used inverse theorems for entropy to prove $\dim_H \mu_{\beta,p} = 1$ for $\beta \in (p^p(1-p)^{1-p},1)$ outside a set Hausdorff dimension zero. Shmerkin [13] used the result of Hochman to prove the theorem stated above.

The challenging question that remains open is to characterized the exceptional values β for which $\mu_{\beta,p}$ is singular. As fare as we know the only known result so far is due to Erdös [3]. Recall here that a Pisot number is an algebraic integer with all its conjugates inside the unite, see [1].

Theorem 1.2 *If $\beta \in (0.5, 1)$ is the reciprocal of a Pisot number $\mu_{\beta,p}$ is singular for all $p \in (0, 1)$*

In fact Erdös proved the theorem for $p = 0.5$ but the prove can by generalized easily. In this paper we will demonstrate that it is quite easy to get the singularity of $\mu_{\beta,p}$ for algebraic numbered $\beta \in (0.5, 1)$ that are not Pisot numbers and some p with $p^p(1-p)^{1-p} < \beta$.

Theorem 1.3 *For all algebraic integers $\beta \in (0.5, 1)$ which full fill an algebraic equation with coefficients in $\{-1, 0, 1\}$ there is an $\mathfrak{p} = \mathfrak{p}(\beta) \in (0, 1/2]$ with $\beta > \mathfrak{p}^{\mathfrak{p}}(1-\mathfrak{p})^{(1-\mathfrak{p})}$ such that $\mu_{\beta,p}$ is singular for all $p < \mathfrak{p}$.*

Examples of algebraic integers for which our theorem applies are the reciprocals of $\sigma_n \in (1, 2)$, given by the dominating root of

$$x^n - x^{n-1} - \cdots - x + 1 = 0, \quad n \geq 4,$$

and the reciprocal of $\nu_n \in (1, 2)$ given by the dominating root of

$$x^n - x^{n-1} - \cdots - x^3 - 1 = 0, \quad n \geq 4.$$

Note that the numbers σ_n are Salem numbers (the conjugates have absolute value no greater than 1, and at least one of them has absolute value exactly 1) and the numbers ν_n are neither Pisot nor Salem numbers, see [1] and also [4]. The mayor challenge that remains is to determine $\mathfrak{p}(\beta)$ for these numbers. Our techniques only give lower bounds, which we compute in the last section for $\beta = \sigma_4^{-1}, \tau_4^{-1}$. The next section in devoted to the proof of Theorem 3.1.

2 Proof of Theorem 4.1

For $\beta \in (0, 1)$ we define a sequence of relations $\sim_n$ on $\{-1, 1\}^{\mathbb{N}}$ by

$$(s_i) \sim_n (t_i) :\Leftrightarrow \sum_{i=1}^{n} s_i \beta^i = \sum_{i=1}^{n} t_i \beta^i.$$

Obviously this are equivalence relations. Let $\mathcal{P}_n$ be the partition of $\{-1, 1\}^{\mathbb{N}}$ induced by $\sim_n$ and let b^p be the Bernoulli measure on $\{-1, 1\}^{\mathbb{N}}$ for $p \in (0, 1)$. Define the entropy of the Partition $\mathcal{P}_n$ with respect to the Bernoulli measure by

$$H_p(\mathcal{P}_n) = -\sum_{P \in \mathcal{P}_n} b^p(P) \log(b^p(P)).$$

Observe that the partition $\mathcal{P}_n \vee \sigma^{-n}\mathcal{P}_m$ is finer than $\mathcal{P}_{n+m}$. Hence by well known properties of the partition entropy, $H_\mu(\mathcal{P}_n)$ is a sub-additive sequence, see [15]. Hence the Garsia entropy

$$G_\beta(p) = \lim_{n \longmapsto \infty} \frac{H_p(\mathcal{P}_n)}{n} = \inf\{\frac{H_p(\mathcal{P}_n)}{n} | n \geq 1\}$$

exists; compare with [5] and [6]. The following lemma is essential for our proof of theorem 1.3.

Lemma 2.1 *For all algebraic integers $\beta \in (0.5, 1)$ which full fill an algebraic equation with coefficients in $\{-1, 0, 1\}$ we have*

$$G_\beta(p) \leq U_\beta(p) < -p\log(p) - (1-p)\log(1-p)$$

where $U_\beta : (0,1) \to \mathbb{R}$ is a continuous function.

Proof. By the assumption the partition $\mathcal{C}_n$ of $\{-1,1\}^{\mathbb{N}}$ into cylinder sets is finer than the partition $\mathcal{P}_n$ if n is large enough. Hence

$$H_p(\mathcal{P}_n) < H_p(\mathcal{C}_n) = -np\log(p) - n(1-p)\log(1-p).$$

$U_\beta(p) := H_p(\mathcal{P}_n)/n$ is the continuous function we are searching for. □

Beside this lemma we will use proposition 3 of [9]:

Proposition 2.1 *For all $p \in (0,1)$ and $\beta \in (0,1)$*

$$\dim_H \mu_{\beta,p} \leq \frac{G_\beta(p)}{\log \beta^{-1}}.$$

where $\dim_H \mu_{\beta,p}$ denotes the Hausdorff dimension of the measure.

Proof of theorem 3.1 Fix β and choose U_β from lemma 2.1. Let

$$\mathfrak{p} = \sup\{\tilde{p} \in (0, 1/2] \mid U_\beta(p) < \log \beta^{-1} \ \forall p \in (0, \tilde{p}]\}$$

If $p < \mathfrak{p}$ we obtain by lemma 2.1 and proposition 3.1

$$\dim_H \mu_{\beta,p} \leq \frac{G_\beta(p)}{\log \beta^{-1}} \leq \frac{U_\beta(p)}{\log \beta^{-1}} < 1,$$

hence $\dim_H \mu_{\beta,p}$ is singular. If $\beta \leq p^p(1-p)^{1-p}$ for some p, we have

$$U_\beta(p) < -p\log(p) - (1-p)\log(1-p) \leq \log \beta^{-1}.$$

By continuity of $U_\beta(p)$ we get $\beta > \mathfrak{p}^{\mathfrak{p}}(1-\mathfrak{p})^{(1-\mathfrak{p})}$. □

3 Computational results

Let $\beta = \sigma_4^{-1}$ where σ_4 is dominating root of

$$x^4 - x^3 - x^2 - x + 1 = 0,$$

compare with the introduction. We want to compute the partition $\mathcal{P}_n$ defined in the last section with respect to β. To this end consider the recursion

$$N_{n+4} = N_{n+3} + N_{n+2} + N_{n+1} - N_0$$

with $N_0 = 1$, $N_1 = 2$, $N_2 = 4$, $N_3 = 8$. Note that

$$\sum_{i=1}^{n} s_i\beta^i = \sum_{i=1}^{n} t_i\beta^i \Leftrightarrow \sum_{i=1}^{n} s_i N_i = \sum_{i=1}^{n} t_i N_i \qquad (\star)$$

We use this relation to compute the partition $\mathcal{P}_{18}$ explicitly using Mathematics. With this we easily compute the Bernoulli measures of the partition elements and get thus an (lengthy) explicit expression for the entropy $h(p) := H_p(\mathcal{P}_{18})$ of the partition. It turns out that $h(p)/(18 \log \sigma_4) < 1$ for $p < 0.29$ hence we obtain by the results of last section:

Proposition 3.1 *For all $p < 0.29$ the measure $\mu_{\sigma_4^{-1},p}$ is singular.*

Note that

$$\beta = \sigma_4^{-1} \approx 0.5806 > 0.29^{0.29} \cdot 0.71^{0.71} \approx 0.5476$$

hence σ_4^{-1} is an exception in the domain of generic absolute continuity given by theorem 1.1. Now let $\beta = \nu_4^{-1}$ where ν_4^{-1} is the dominating root of

$$x^4 - x^3 - 1 = 0.$$

We consider the recursion

$$N_{n+4} = N_{n+3} + N_0$$

with $N_0 = 1$, $N_1 = 2$, $N_2 = 4$, $N_3 = 8$ and again get the relation $(\star)$. This allows us to compute $\mathcal{P}_{18}$ and $h(p)$ explicitly using Mathematica. It turns out that $h(p)/(18 \log \nu_4) < 1$ for $p < 0.13$ hence:

Proposition 3.2 *For all $p < 0.13$ the measure $\mu_{\nu_4^{-1},p}$ is singular.*

We have

$$\beta = \nu_4^{-1} \approx 0.7244 > 0.13^{0.13} \cdot 0.87^{0.87} \approx 0.695$$

and ν_4^{-1} is an exception in theorem 1.1.

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