On the Hausdorff dimension of fractals given by certain expansions of real numbers

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Abstract

We transfer classical results on the Hausdorff dimension of b-adic and continued fraction expansions of real numbers to another expansion.

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1 Introduction

Dimension theory comes into the study of real numbers if we consider b-adic expansions

$$\sum_{k=1}^{\infty} n_k b^{-k},$$

where the digits n_k comes from some non-empty subset $A \subseteq \{0, 1, \ldots, b-1\}$ of all digits. The set $D(A)$ of such numbers is a fractal, its Hausdorff dimension is given by

$$\dim_H D(A) = \frac{\log(\sharp A)}{\log b}.$$

This result goes back to the work of Hausdorff [6]. If we prescribe the frequencies $(p_0, \ldots p_{b-1}) \in [0, 1]^b$ of digits that appear in the expansion, we get a fractal $D(p_0, \ldots p_{b-1})$ with Hausdorff dimension

$$\dim_H D(p_0, \ldots p_{b-1}) = \frac{-\sum_{i=0}^{b-1} p_i \log(p_i)}{\log b}.$$

This classical result is due to Besicovitch [1] for dyadic expansions and to Eggleston in the general form [2].

Now consider the continued fraction expansion of reals from $(0, 1]$ in $\mathbb{N}^{\mathbb{N}}$,

$$\frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \cdots}}},$$

with prescribed digits d_k coming from a finite set. We again get a fractal, and the Hausdorff dimension of this set was intensively studied since the work of Jarnik [7], who

proved that the dimension of the set E_n of continued fractions with digits in $\{1, \dots, n\}$ can be estimated by

$$1 - \frac{4}{n \log 2} \leq \dim_H E_n \leq 1 - \frac{1}{8n \log n}$$

for $n > 8$. Nowadays we have an efficient non-trivial algorithm to calculate the dimension of E_n. In particular we have

$$\dim_H E_2 = 0.5312805062772\dots.$$

see [8] a references there in.

In this paper we consider the expansion of the reals form $(0, 1]$ in $\mathbb{N}^{\mathbb{N}}$ given by

$$\sum_{k=1}^{\infty} 2^{-(n_1 + n_2 + \dots + n_k)}$$

and ask for the Hausdorff dimension of fractals related to this expansion. This expansion is induced by expanding Markov-Renyi maps, see [13].

In the next section we introduce our notations and prove results on the Hausdorff dimension of such expansion with prescribed digits. In the last section we state and prove our results for prescribed frequencies of the digits in such expansions.

2   Results for prescribed digits

Consider the set of infinite sequences of natural numbers,

$$\mathbb{N}^{\mathbb{N}} = \{(n_k) | n_k \in \mathbb{N}\}$$

and the map $\pi : \mathbb{N}^{\mathbb{N}} \longmapsto (0, 1]$ given by

$$\pi(n_k) = \sum_{k=1}^{\infty} 2^{-(n_1 + n_2 + \dots + n_k)}.$$

The sum converges since $2^{-(n_1 + n_2 + \dots + n_k)} < 2^{-k}$, and we have:

Proposition 2.1 Π *is a bijection.*

Proof.   Represent the numbers form $(0, 1]$ using their binary expansions. With the convention that there is no tail of zeros in the expansion, this representation is unique. We have:

$$\pi(n_k) = 0.\underbrace{0\dots01}_{n_1}\underbrace{0\dots01}_{n_2}\underbrace{0\dots01}_{n_3}\dots$$

Now we see the π is onto, i.e., invertible, since the representation on the right hand side can be arbitrarily prescribed. invertible, since the representation on the right hand side

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is unique. $\square$

So π gives us a metric d on $\mathbb{N}^{\mathbb{N}}$, via

$$d(n_k, m_k) = |\pi(n_k) - \pi(m_k)|,$$

and we see that $\pi(n_k) \in \mathbb{Q}$ if and only if the sequence (n_k) has periodic tails. Given a subset $A \subseteq \mathbb{N}$ we are interested in the set of numbers having a representation in $\mathbb{N}^{\mathbb{N}}$ with entries coming from A, this is

$$\pi(A) = \Pi(A^{\mathbb{N}}).$$

The set $\Pi(A)$ has a fractal geometry, thus we introduce Hausdorff dimension here. Recall [4, 12] that the d-dimensional Hausdorff measure of a set $A \subseteq \mathbb{R}$ is

$$\mathfrak{H}^d(A) = \lim_{\epsilon \longmapsto 0} \inf \{ \sum_{i=1}^{\infty} \mathrm{diam}(C_i)^d | A \subseteq \bigcup_{i=1}^{\infty} C_i, \ \mathrm{diam}(C_i) < \epsilon \}.$$

The Hausdorff dimension of A is given by

$$\dim_H A = \sup\{d | \mathfrak{H}^d(A) = \infty\} = \inf\{d | \mathfrak{H}^d(A) = 0\}.$$

Now we are able to state our main result on $\dim_H \Pi(A)$.

Theorem 2.1 *For $A \subseteq \mathbb{N}$ let $d \in [0, 1]$ be the solution of*

$$\sum_{n \in A} 2^{-nd} = 1.$$

We have $\dim_H \Pi(A) = d$.

Proof. For $n \in A$ consider the contraction $T_n : (0, 1] \longmapsto (0, 1]$ given by

$$T_n x = \left(\frac{1}{2}\right)^n (x + 1).$$

We have

$$\bigcup_{n_0=1}^{\infty} T_{n_0}(\Pi(A)) = \bigcup_{n_0=1}^{\infty} T_{n_0} \left\{ \sum_{k=1}^{\infty} 2^{-(n_1+n_2+\ldots+n_k)} | (n_k) \in A^{\mathbb{N}} \right\}$$

$$= \bigcup_{n_0=1}^{\infty} \left\{ \sum_{k=0}^{\infty} 2^{-(n_0+n_1+n_2+\ldots+n_k)}) | (n_k) \in A^{\mathbb{N}} \right\}$$

$$= \left\{ \sum_{k=0}^{\infty} 2^{-(n_0+n_1+n_2+\ldots+n_k)} | (n_k) \in A^{\mathbb{N}}, \ n_0 \in A \right\} = \Pi(A).$$

This means that $\Pi(A)$ is the attractor of the linear iterated function system $\{T_n | n \in A\}$ on $(0, 1]$, see [6] for finite sets A and [5] for infinite sets.

This systems obviously fulfills the strong open set condition for the open interval $(0,1)$. This means $T_m((0,1)) \subseteq (0,1)$ with

$$T_{m_1}((0,1)) \cap T_{m_2}((0,1)) = \emptyset \text{ for } m_1 \neq m_2$$

and $(0,1) \cap \Pi(A) \neq \emptyset$. The dimension formula in our result is the classical Moran formula. If A is finite the result directly follows from the work of Moran [10]. If A is infinite it follows from theory of infinite iterated function systems see theorem 3.11 of [5] or [11] for a more general approach. $\square$

Our main theorem has many interesting applications:

Corollary 2.1 *For $n \geq 2$ we have $\dim_H \Pi(\{1, \dots n\}) = \log_2(s)$, where $s > 1$ is the solution of*

$$s^n - s^{n-1} - \dots - s = 1.$$

In particular we have

$$\dim_H \Pi(\{1,2\}) = \log_2(\frac{\sqrt{5}+1}{2}) = 0.6942419136\ldots.$$

Proof. For $d > 0$ we have

$$\sum_{i=1}^{n} 2^{-id} = \frac{2^{-d} - 2^{-d(n+1)}}{1 - 2^{-d}} = 1 \Leftrightarrow 2 \cdot 2^{-d} - 2^{-d(n+1)} = 1 \Leftrightarrow 2 \cdot 2^{dn} - 2^{d(n+1)} = 1.$$

Substituting $s = 2^d$ gives $2 \cdot s^n - s^{n+1} = 1$. Hence we get

$$(s-1)(s^n - s^{n-1} - \dots - s - 1) = -2s^n + s^{n+1} + 1 = 0,$$

leading to the result since $s \neq 1$. For $n = 2$ we obtain $s^2 - s - 1 = 0$. $\square$

Corollary 2.2 *Given $b \in \mathbb{N}$, let $A_b = \{(nb)|n \in \mathbb{N}\}$. We have $\dim_H \Pi(A_b) = 1/b$.*

Proof. For $d > 0$ we have

$$\sum_{n=1}^{\infty} 2^{-nbd} = \frac{1}{1 - 2^{-bd}} - 1 = 1 \Leftrightarrow d = 1/b.$$

$\square$

Corollary 2.3 *Given $b \geq 2$ and $c \in \{1, \dots, b-1\}$, let $A_{b,c} = \{(nb+c)|n \in \mathbb{N}_0\}$. We have $\dim_H \Pi(A_{b,c}) = \log_2(1/r)$ where $r \in (0,1)$ is the solution of*

$$r^c + r^b = 1.$$

In particular we have

$$\dim_H \Pi(A_{2,1}) = \dim_H \Pi(\{1,2\}).$$

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Proof.

$$\sum_{n=0}^{\infty} 2^{-(nb+c)d} = \frac{2^{-dc}}{1 - 2^{-bd}} = 1 \Leftrightarrow 2^{-cd} + 2^{-bd} = 1$$

Substituting $r = 2^{-d}$ gives $r^c + r^b = 1$ leading to the result. For $b = 2$ and $c = 1$ we have $r^2 + r^1 = 1$. And if we set $s = 1/r$, s fulfills $s^2 - s - 1 = 0$. $\qquad\square$

3 Results for prescribed frequencies

Let $\mathfrak{p} = (p_i)$ be a probability distribution on $\mathbb{N}$ which is supported by a finite set A, this means

$$\sum_{i \in A} p_i = 1 \qquad \text{and} \qquad p_i = 0 \quad \text{for} \quad i \notin A.$$

The expected value of $\mathfrak{p}$ is

$$E(\mathfrak{p}) = \sum_{i \in A} i p_i$$

and the entropy is given by

$$H(\mathfrak{p}) = -\sum_{i \in A} p_i \log(p_i).$$

We are interested in the set of numbers in $\Pi(A)$ with frequency p_i of i in the expansion $\pi^{-1}(x)$, this is

$$A(\mathfrak{p}) = \{\pi(n_k) \in \Pi(A) | \lim_{k \longmapsto \infty} \frac{\sharp\{l | n_l = i, \quad l = 1, \dots, k\}}{k} = p_i\}.$$

Theorem 3.1 *For a probability distribution $\mathfrak{p}$ on $\mathbb{N}$ which is supported by a finite set A we have*

$$\dim_H A(\mathfrak{p}) = \frac{H(\mathfrak{p})}{\log(2)E(\mathfrak{p})}.$$

Proof. Consider the Borel probability measure $\mathfrak{m}$ on $\mathbb{N}^{\mathbb{N}}$ which is the infinite product of the probability measure given by $\mathfrak{p}$ on $\mathbb{N}$. Project this measure to $(0,1]$ using π

$$\mu = \pi(\mathfrak{m}) = \mathfrak{m} \circ \pi^{-1}.$$

By the law of large numbers we have

$$\lim_{k \longmapsto \infty} \frac{\sharp\{l | n_l = i, \quad l = 1, \dots, k\}}{k} = p_i \quad \forall i \in A$$

if we choose the digits of a sequence (n_k) randomly from A according to $\mathfrak{p}$. Hence this equation holds for $\mathfrak{m}$-almost all sequences in $\mathbb{N}^{\mathbb{N}}$. As a consequence we have $\mu(A(\mathfrak{p})) = 1$. Now define intervals contained in $(0,1]$ by

$$I_{n_1 \dots n_k} = T_{n_k} \circ \dots \circ T_{n_1}((0,1])$$

for $n_i \in A$ where the maps T_n were defined in section 2. For $x \in \Pi(A)$ let $I_{n_1 \dots n_k}(x)$ be the unique interval containing x. Note that $\pi^{-1}(x) = (n_1, \dots, n_k, \dots)$. Furthermore let

$$\sharp_i(x|k) = \sharp\{l|n_l = i, \quad l = 1, \dots, k\}.$$

For the length of the intervals we obtain

$$\log(|I_{n_1 \dots n_k}(x)|) = \log(2^{-(n_1 + n_2 + \dots + n_k)}) = -\log(2) \sum_{i \in A} i \sharp_i(x|k)$$

and for the measure of the intervals we get

$$\log \mu(I_{n_1 \dots n_k}(x)) = \log(\prod_{j=1}^{k} p_{n_j}) = \sum_{i \in A} \sharp_i(x|k) \log p_i$$

by the definition of μ. If $x \in A(\mathfrak{p})$ we have

$$\lim_{k \to \infty} \frac{\sharp_i(x|k)}{k} = p_i$$

for all $i \in A$ by the definition of this set. Thus we have

$$\lim_{k \to \infty} \frac{1}{k} \log \frac{\mu(I_{n_1 \dots n_k}(x))}{|I_{n_1 \dots n_k}(x)|^s} = \lim_{k \to \infty} \frac{1}{k} (\sum_{i \in A} \sharp_i(x|k) \log p_i + s \log(2) \sum_{i \in A} i \sharp_i(x|k))$$

$$= -H(X) + s \log(2) E(X).$$

This implies for all $x \in A(\mathfrak{p})$

$$\lim_{k \to \infty} \frac{\mu(I_{n_1 \dots n_k}(x))}{|I_{n_1 \dots n_k}(x)|^s} = \{ \begin{matrix} 0 & s < d \\ \infty & s > d \end{matrix}$$

with

$$d = \frac{H(\mathfrak{p})}{\log(2) E(\mathfrak{p})}.$$

By the local mass distribution principle, see Proposition 4.9 of [4], we have $\mathfrak{H}^s(A(\mathfrak{p})) = \infty$ for $s < d$ and $\mathfrak{H}^s(A(\mathfrak{p})) = 0$ for $s > d$. This implies $\dim_H(A(\mathfrak{p})) = d$. $\qquad \square$

Theorem 3.1 has following corollary.

Corollary 3.1 *If $\mathfrak{p}$ has finite support, then $\dim_H A(\mathfrak{p}) < 1$.*

Proof. The maximum of the dimension function $d(\mathfrak{p}) = \dim_H A(\mathfrak{p})$ is given by the solution d of

$$\sum_{i \in A} 2^{-id} = 1$$

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for $p_i = 2^{-id}$. This solution is smaller than 1. $\square$

On the other hand there exists a distribution on $\mathbb{N}$ which has full dimension. Namely, define the set of numbers in $\mathcal{A} \subseteq (0, 1]$ with frequency $1/2^i$ of i in the expansion $\pi^{-1}(x)$ by

$$\mathcal{A} = \{\pi(n_k)|\lim_{k \longmapsto \infty} \frac{\sharp\{l|n_l = i, \quad l = 1, \ldots, k\}}{k} = \frac{1}{2^i} \quad \forall i \in \mathbb{N}\}.$$

Theorem 3.2 *We have* $\dim_H \mathcal{A} = 1$.

Proof. Consider a Borel probability measure $\mathfrak{m}$ on $\mathbb{N}^{\mathbb{N}}$ which is the infinite product of an arbitrary probability distribution $\mathfrak{p}$ on $\mathbb{N}$. Project this measure to $(0, 1]$ using π to obtain a measure μ on $(0, 1]$. Applying theorem A of [9] to the infinite iterated function system $\{T_n|n \in \mathbb{N}\}$ we obtain that the Hausdorff dimension of μ,

$$\dim_H(\mu) = \inf\{\dim_H A|\mu(A) = 1\},$$

is given by

$$\dim_H \mu = \frac{\sum_{i=1}^{\infty} p_i \log(p_i)}{\sum_{i=1}^{\infty} p_i \log(2^{-i})}.$$

Now inserting $p_i = 1/2^i$ we get $\dim_H \mu = 1$. But by the law of large numbers $\mu(\mathcal{A}) = 1$. Hence $\dim_H \mathcal{A} \geq 1$, which obviously implies the result. $\square$

To prove an upper bound on $\dim_H A(\mathfrak{p})$ for arbitrary probability sequences $\mathfrak{p}$ seems to be a hard problem, the proof of Theorem 3.1 does not work. In probabilistic terminology, if we have a sequence of unbounded random variables, the expected value and entropy do not have to converge to the expected value and entropy of the limit.

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