The weighted Moran formula for the dimension of generic McMullen-Bedford carpets

J. Neunhäuserer

University of Applied Science Berlin
Luxemburger Strae 10
13353 Berlin, Germany
neunchen@aol.com

Abstract

We introduce a weighted version of the classical Moran formula for the dimension of certain self-affine sets in the plane.
MSC 2000: Primary 28A80, Secondary 28D20
Key-words: Self-affine carpets, Self-similar measures, Hausdorff dimension, Minkowsky dimension

1 Introduction

For the Hausdorff and Minkowsky dimension D of self-similar sets fulfilling the open set condition we have the famous Moran formula,

$$\sum {\tau_i}^D = 1,$$

where τ_i are the contraction rates of the similitudes, see Moran [13]. Carpets that are aligned self-affine sets in the plane where studied by McMullen [12] and independently by Bedford [2]. In the case that the contraction rates of the affinities are rational, formulas for the Hausdorff and Minkowsky dimension of the carpets, that do not coincide in this general, were given. For an introduction to the dimension theory and especially the definition of Hausdorff and Minkowsky dimension we recommend the book of Falconer [3] or the book of Pesin [9].
We consider here a certain class of aligned self-affine carpets which have irrational contraction rates in general. We proof that generically, in the sense of Lebesgue measure on the parameter space, a weighted version of the Moran formula,

$$\sum \beta_i {\tau_i}^{D-1} = 1$$

gives the Hausdorff and the Minkowsky dimension D of the carpets. Here τ_i are the strong contraction rates and β_i are the weak contraction rates of the affinities.
To get this result we proceed in three steps. First we proof by an elementary covering argument that D is an upper bound of the Minkowsky dimension of the carpets. In the second steps we develop the dimension theory for Bernoulli measures on the carpets. We use general results from the dimension theory of dynamical systems to get a dimension formula for these measures in terms of Lyapunov Exponents and the dimension of transversal measures, [6], [1]. In fact these transversal measures are itself self-similar Bernoulli measures on the real line. In the third step we use results on absolute continuity of these self-similar measures from [8] to proof that the dimension of well weighted

Bernoulli measures on self-affine carpets have dimension D.
The rest of the paper is organized as follows: In section one we introduce our notations and main results, than we discuss two examples. In section three we proof, that the Minkowsky dimension of the carpets is bounded by D. In section four we develop the dimension theory of Bernoulli measures on the carpets. We conclude with an application of the theory of self-similar measures on the real line in section five.

2 Notations and main results

Consider vectors of contractions $\mathfrak{b} = (\beta_1, \dots, \beta_n) \in (0,1)^n$, $\mathfrak{t} = (\tau_1, \dots, \tau_n) \in (0,1)^n$ and vectors of translations $\mathfrak{d} = (d_1, \dots, d_n) \in \mathbb{R}^n$, $\mathfrak{e} = (e_1, \dots, e_n) \in \mathbb{R}^n$ and corresponding affine maps on the plan $\mathbb{R}^2$ given by

$$f_i(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} \beta_i x + d_i \\ \tau_i y + e_i \end{pmatrix}$$

for $i = 1 \dots n$. Let $\mathfrak{i} = (\mathfrak{b}, \mathfrak{t}, \mathfrak{d}, \mathfrak{e})$ be the vector of all parameters. Throughout the paper we assume

$$\sum_{i=1}^{n} \beta_i > 1 \quad \text{and} \quad \sum_{i=1}^{n} \tau_i < 1.$$

Moreover we assume, that the images $f_i(R)$ of some rectangle R do not intersect, that is:

$$f_i(R) \cap f_j(R) = \emptyset \quad \text{for} \quad i \neq j.$$

Define the set of all parameters $\mathfrak{I}$ accordingly. These assumption are necessary to apply the weighted Moran formula.
From [4] we know that there is a unique compact attractor $\Lambda_{\mathfrak{i}}$ of the iterated function system given by the contractions $\{f_i | i = 1, \dots, n\}$ fulfilling

$$\Lambda_{\mathfrak{i}} = \bigcup_{i=1}^{n} f_i(\Lambda_{\mathfrak{i}}).$$

For all $\mathfrak{i} \in \mathfrak{I}$ this is an aligned self- affine set in the plan. Now consider a probability vector $\mathfrak{p} = (p_1, \dots, p_n) \in (0,1)^n$ with $\sum_{i=1}^{n} p_i = 1$. We know [4] that there is a unique Borel probability measure $\mu_{\mathfrak{p}}$ on $\Lambda_{\mathfrak{i}}$ fulfilling

$$\mu_{\mathfrak{p}} = \sum_{i=1}^{n} p_i f_i(\mu_{\mathfrak{p}}) = \sum_{i=1}^{n} p_i (\mu_{\mathfrak{p}} \circ f_i^{-1}).$$

Now consider for a moment the projection of the system onto the first coordinate axis,

$$\hat{f}_i(x) = \beta_i x + d_i,$$

with projected measures $\hat{\mu}_{\mathfrak{p}}$ fulfilling

$$\hat{\mu}_{\mathfrak{p}} = \sum_{i=1}^{n} p_i f_i(\hat{\mu}_{\mathfrak{p}}) = \sum_{i=1}^{n} p_i (\hat{\mu}_{\mathfrak{p}} \circ f_i^{-1}).$$

Our first result uses properties of the projected system to find the dimension of the carpets $\Lambda_{\mathfrak{i}}$ using the the weighted Moran formula.

Theorem 2.1 *For $\mathfrak{i} \in \mathfrak{I}$ let D be the solution of*

$$\sum \beta_i {\tau_i}^{D-1} = 1$$

and let $\mathfrak{p} = (\beta_1 {\tau_1}^{D-1}, \ldots, \beta_n {\tau_n}^{D-1})$. If $\hat{\mu}_{\mathfrak{p}}$ is absolutely continues the Hausdorff and Minkowsky dimension of the self-affine carpet $\Lambda_{\mathfrak{i}}$ is given by

$$\dim_H \Lambda_{\mathfrak{i}} = \dim_M \Lambda_{\mathfrak{i}} = D.$$

To establish absolute continuity of $\hat{\mu}_{\mathfrak{p}}$ we use the transversallity technique, compare [8] and references there in. For $b > 0$ consider the space of analytic functions with $f(0) = 1$ and coefficients in the interval $[-r, r]$, i.e.

$$\mathcal{F}_b = \{f(x) = 1 + \sum_{k=1}^{\infty} r_k x^k | r_k \in [-r, r]\}$$

and let

$$\mathrm{trans}(r) = \min\{x > 0 | \exists f \in \mathcal{F}_r \text{ with } f(x) = f'(x)\}.$$

Given an arbitrary $\epsilon > 0$ there is $\rho > 0$ such that each function $f \in \mathcal{F}_r$ crosses the x-axis transversely with slope in $[-\rho, \rho]$ on the interval of transverselity $[0, \mathrm{trans}(r) - \epsilon]$. We know from [10] and [11]that

$$\mathrm{trans}(1) = 0.64913..., \quad \mathrm{trans}(2) = 0,5, \quad \mathrm{trans}(3) = 0.42772...,$$

$$\mathrm{trans}(r) \geq (\sqrt{b} + 1)^{-1} \text{ for } b \in [1, 3 + \sqrt{8}],$$
$$\mathrm{trans}(r) = (\sqrt{b} + 1)^{-1} \text{ for } b \in [3 + \sqrt{8}, \infty).$$

In our context we let

$$r(\mathfrak{b}, \mathfrak{d}) = \frac{\max\{\beta_i d_j | d_j > 0\} + \max\{-\beta_i d_j | d_j < 0\}}{\min_{i \neq j} |d_i - d_j|}.$$

and $\mathrm{trans}(\mathfrak{b}, \mathfrak{d}) = \mathrm{trans}(r(\mathfrak{b}, \mathfrak{d}))$. Furthermore let

$$\mathfrak{B}(\mathfrak{d}) = \bigcup_{\mathfrak{a} \in (0,1)^n} \{(v\alpha_1, \ldots v\alpha_n) | v \in ((\sum_{1=1}^{n} \alpha_i)^{-1}, \mathrm{trans}(\mathfrak{a}, \mathfrak{d}))\}$$

be the parameter domain of transversallity. With this notations we state our main result

Theorem 2.2 *Let $\mathfrak{i} = (\mathfrak{b}, \mathfrak{t}, \mathfrak{d}, \mathfrak{e}) \in \mathfrak{I}$. For almost all $\mathfrak{b} \in \mathfrak{B}(\mathfrak{d})$ (in the sense of Lebesgue measure on all lines through the origin) the Hausdorff and Minkowsky dimension of the self-affine carpets $\Lambda_{\mathfrak{i}}$ are given by*

$$\dim_H \Lambda_{\mathfrak{i}} = \dim_M \Lambda_{\mathfrak{i}} = D$$

where D be the solution of

$$\sum \beta_i {\tau_i}^{D-1} = 1.$$

We conjecture that the assumption of transversallity is not necessary and that the upper bound in our theorem is hence not sharp.

3 Examples

Consider self affine sets Λ_v given by the following transformation

$$f_1(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} vx \\ 1/4y \end{pmatrix}, \quad f_2(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} 2/3vx+1 \\ 1/4y+1 \end{pmatrix}, \quad f_3(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} 3/4vx+2 \\ 1/4y+2 \end{pmatrix}.$$

For $v > 12/29$ the sum of the contraction rates in the first coordinate direction is bigger than one and the sum of contraction rates in the second direction is less than one. Moreover the images of the maps are separated for some rectangle, by the choice of translations in the second coordinate direction. In this situation our techniques apply. The upper bound on the transversallity interval is 0.5 (since $r(\mathfrak{b}, \mathfrak{d}) = 2$). By theorem 2.2 we have for almost all $v \in (12/29, 0.5)$

$$\dim_H \Lambda_v = \dim_M \Lambda_v = \frac{\log(12/(29v))}{\log 4} + 1.$$

As far as we know results of this type are completely new. So let us discuss a second example. Consider now the self affine sets Λ_v given by the transformation

$$f_1(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} 1/2vx \\ 1/4y \end{pmatrix}, f_2(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} 1/2vx+1 \\ 1/4y \end{pmatrix}, f_3(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} vx+1 \\ 1/5y+1 \end{pmatrix}.$$

Here for $v > 1/2$ our result applies. The upper bound on the transversallity interval is 0.649 (since $r(\mathfrak{b}, \mathfrak{d}) = 1$). By theorem 2.2 the Minkowsky and Hausdorff dimension of the carpet Λ_v is given by the solution D of

$$(1/5)^{D-1} + (1/4)^{D-1} = 1/v$$

for almost all $v \in (0.5, 0.649)$. The reader might constructed other examples on demand.

4 An estimate on the Minkowsky dimension of self-affine carpets

We proof here an upper bound on the Minkowsky dimension of self-affine carpets. The argument we use seems to be nowadays folklore in the dimension theory. We just construct a cover by squares and estimate the numbers of covering elements.

Proposition 4.1 *For all parameter values $\mathfrak{i} \in \mathfrak{I}$ the Minkowsky dimension of the self-affine carpet $\Lambda_{\mathfrak{i}}$ is bounded from above by the solution D of*

$$\sum_{i=1}^{n} \beta_i {\tau_i}^{D-1} = 1.$$

Proof. Choose an aligned rectangle R with side length s_1 and s_2 such that $f_i(R) \subseteq R$ for all $i = 1, \ldots n$. For a sequence $(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$ the set

$$f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_k}(R)$$

is an aligned rectangle with side length $\beta_{i_1}\beta_{i_2}\ldots\beta_{i_n}s_1$ and $\tau_{i_1}\tau_{i_2}\ldots\tau_{i_n}s_2$. Note that this rectangle can be covered by

$$\lceil\frac{\beta_{i_1}\beta_{i_2}\ldots\beta_{i_n}s_1}{\tau_{i_1}\tau_{i_2}\ldots\tau_{i_n}s_2}\rceil$$

squares of side length $\tau_{i_1}\tau_{i_2}\ldots\tau_{i_n}s_2$, where $\lceil x\rceil$ denotes the smallest integer bigger than x.

Given $r>0$ define a set of finite sequences by

$$\Sigma(r)=\{(i_1,\ldots,i_k)|r\min\{\tau_i\}\le\tau_{i_1}\tau_{i_2}\ldots\tau_{i_n}s_2\le r\}$$

and let $N(r)$ be the minimal cardinality of squares of the length r needed to cover the set Λ_{i}. Since

$$\Lambda_{\mathsf{i}}\subseteq\bigcup_{(i_1,\ldots,i_k)\in\Sigma(r)}f_{i_1}\circ\ldots f_{i_k}(R)$$

we have

$$N(r)\le\sum_{(i_1,\ldots,i_k)\in\Sigma(r)}\lceil\frac{\beta_{i_1}\beta_{i_2}\ldots\beta_{i_n}s_1}{\tau_{i_1}\tau_{i_2}\ldots\tau_{i_n}s_2}\rceil$$

and hence

$$N(r)r^D\le\sum_{(i_1,\ldots,i_k)\in\Sigma(r)}\lceil\frac{\beta_{i_1}\beta_{i_2}\ldots\beta_{i_n}s_1}{\tau_{i_1}\tau_{i_2}\ldots\tau_{i_n}s_2}\rceil(\tau_{i_1}\tau_{i_2}\ldots\tau_{i_n}s_2\min\{\tau_i\}^{-1})^D$$

$$\le\min\{\tau_i\}^{-D}\sum_{(i_1,\ldots,i_k)\in\Sigma(r)}(\beta_{i_1}\beta_{i_2}\ldots\beta_{i_n}s_1(\tau_{i_1}\tau_{i_2}\ldots\tau_{i_n}s_2)^{D-1}+(\tau_{i_1}\tau_{i_2}\ldots\tau_{i_n}s_2)^D)$$

$$=\min\{\tau_i\}^{-D}(s_1s_2^{D-1}\sum_{(i_1,\ldots,i_k)\in\Sigma(r)}(\beta_{i_1}\beta_{i_2}\ldots\beta_{i_n}(\tau_{i_1}\tau_{i_2}\ldots\tau_{i_n})^{D-1}+s_2^D\sum_{(i_1,\ldots,i_k)\in\Sigma(r)}(\tau_{i_1}\tau_{i_2}\ldots\tau_{i_n})^D).$$

Let $\ell(r)$ be the maximal length of a sequence in $\Sigma(r)$. Using the Binomial theorem we have

$$N(r)r^D\le\min\{\tau_i\}^{-D}(s_1s_2^{D-1}(\sum_{i=1}^n\beta_i\tau_i^{D-1})^{\ell(r)}+s_2^D(\sum_{i=1}^n\tau_i^D)^{\ell(r)})$$

$$\le\min\{\tau_i\}^{-D}(s_1s_2^{D-1}+s_2^D).$$

Hence we can estimate the Minkowsky dimension of Λ_{i},

$$\dim_M\Lambda_{\mathsf{i}}=\lim_{r\longmapsto 0}\frac{\log(N(r))}{-\log(r)}\le\lim_{r\longmapsto\infty}\frac{\log(r^{-D})+\log(\min\{\tau_i\}^{-D}(s_1s_2^{D-1}+s_2^D))}{-\log(r)}=D.$$

□

5 The dimension Bernoulli measures on self-affine carpets

In this section we apply the general dimension theory of dynamical systems in order to determine the dimension Bernoulli measures on self-affine carpets in terms of self-similar

Bernoulli measures on the real line. To this end define a map f on $\mathbb{R}^2 \times [0,1]$ by

$$f(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} \beta_i x + d_i \\ \tau_i y + e_i \\ nz - (i-1)/n \end{pmatrix}$$

if $z \in [(i-1)/n, i/n)$ for $i = 1, \dots, n-1$ or $z \in [(n-1)/n, 1]$ for $i = n$. f is contracting in the first coordinate directions and expanding in the last. Obviously it has global attractor given by

$$\bar{\Lambda}_{\mathfrak{i}} = \Lambda_{\mathfrak{i}} \times [0,1],$$

since the contractions are given by the maps f_i from our construction of self-affine carpets $\Lambda_{\mathfrak{i}}$.

Given the Bernoulli measure $\mu_{\mathfrak{p}}$ for a probability vector $\mathfrak{p}$ on the carpet $\Lambda_{\mathfrak{i}}$ we introduce Borel probability measures on $\bar{\Lambda}_{\mathfrak{i}}$ by

$$\bar{\mu}_{\mathfrak{p}} = \mu_{\mathfrak{p}} \times \ell_{\mathfrak{p}}$$

where $\ell_{\mathfrak{p}}$ is the self-similar Bernoulli measure on $[0,1]$ with $\ell_{\mathfrak{p}}((i-1)/n, i/n)) = p_i$. Note that the projection of $\bar{\mu}_{\mathfrak{p}}$ on the first coordinate axis are the self-similar Bernoulli measures $\hat{\mu}_{\mathfrak{p}}$ introduces in second two.

For $h(z) = nz - (i-1)/n$ the system $([0,1], h, \ell_{\mathfrak{p}})$ is well known to be ergodic, moreover the Bernoulli measures $\bar{\mu}_{\mathfrak{p}}$ on the stable slides are ergodic, see [5]. By the product property of the measure $\mu_{\mathfrak{p}}$ the system $(\bar{\Lambda}_{\mathfrak{i}}, f, \bar{\mu}_{\mathfrak{p}})$ is ergodic as well. The metric entropy of this ergodic system is given by

$$h(\mathfrak{p}) = -\sum_{i=1}^{n} p_i \log p_i.$$

We now introduce stable and strong stable direction for f by

$$\mathbb{E}^s = \mathbb{R} \times \mathbb{R} \times \{0\} \qquad \mathbb{E}^{ss} = \{x\} \times \mathbb{R} \times \{z\}$$

We have the following lemma on the Lyapunov exponents of the system $(\bar{\Lambda}_{\mathfrak{i}}, f, \bar{\mu}_{\mathfrak{p}})$

Lemma 5.1 *For $\bar{\mu}_{\mathfrak{p}}$-almost all $(x,y,z) \in \bar{\Lambda}_{\mathfrak{i}}$ we have*

$$\lim_{n \longrightarrow \infty} \frac{1}{n} \log ||D_{(x,y,z)} f^n \overrightarrow{v}|| = \left\{ \begin{array}{lll} \Xi(\mathfrak{b}, \mathfrak{p}) & if & \overrightarrow{v} \in \mathbb{E}^s \backslash \mathbb{E}^{ss} \\ \Xi(\mathfrak{t}, \mathfrak{p}) & if & \overrightarrow{v} \in \mathbb{E}^{ss} \end{array} \right.$$

where

$$\Xi(\mathfrak{b}, \mathfrak{p}) = -\sum_{i=1}^{n} p_i \log \beta_i \quad \text{and} \quad \Xi(\mathfrak{t}, \mathfrak{p}) = -\sum_{i=1}^{n} p_i \log \tau_i.$$

Proof. This follows directly from Birkhoff's ergodic theorem, see the proof of proposition 5.1 in [8]. □

To apply the dimension theory of dynamical systems ergodicity and the existence of Lyapunov exponents is not enough. We have to guarantee the existence of Lyapunov charts associated with the Lyapunov exponents almost everywhere. This means that the set of points that does not approach the singularity S of the dynamical with exponential rate has full measure, see [14].

Lemma 5.2 *For all* $\epsilon > 0$

$$\bar{\mu}_{\mathfrak{p}}(\{(x,y,z) \in \Lambda_{\mathfrak{i}} | \exists l > 0 \ \forall n > 0 \ : \ d(f^n(x,y,z), S) > (1/l)e^{-\epsilon n}\}) = 1.$$

Proof. This is proofed using an argument from symbolic dynamics, see the proof of proposition 5.2 in [8] □

By lemma 5.1 and lemma 5.2 our systems fall into the class of generalized hyperbolic attractors in the sense of Schmeling and Troubetzkoy [14]. Usually the dimension theory of ergodic measures is stated in the context of C^2-diffeomorphisms, but invertibility and the existence of Lyapunov exponents and charts almost everywhere is enough to apply this theory. The following proposition is called the *Ledrappier-Young formula*, it follows directly from [1] and theorem C' of [6].

Proposition 5.1

$$\dim \mu_{\mathfrak{p}} = \frac{h(\mathfrak{p})}{\Xi(\mathfrak{t},\mathfrak{p})} + (1 - \frac{\Xi(\mathfrak{b},\mathfrak{p})}{\Xi(\mathfrak{t},\mathfrak{p})}) \dim \hat{\mu}_{\mathfrak{p}}$$

With the help of proposition 5.1 and proposition 4.1 we are now able to proof theorem 2.1.

Proof of Theorem 2.1 Let $\mathfrak{i} \in \mathfrak{I}$, D be the solution of

$$\sum_{i=1}^{n} \beta_i {\tau_i}^{D-1} = 1$$

and $\mathfrak{p} = (\beta_1 {\tau_1}^{D-1}, \dots, \beta_n {\tau_n}^{D-1})$. Assume that $\hat{\mu}_{\mathfrak{p}}$ is absolutely continues. By definition this implies $\dim \hat{\mu}_{\mathfrak{p}} = 1$. By proposition 5.1 the dimension of the Bernoulli measures on the carpets $\Lambda_{\mathfrak{i}}$ is hence given by:

$$\dim \mu_{\mathfrak{p}} = \frac{\sum_{i=1}^{n} \beta_i {\tau_i}^{D-1} \log(\beta_i {\tau_i}^{D-1})}{\sum_{i=1}^{n} \beta_i {\tau_i}^{D-1} \log(\tau_i)} + (1 - \frac{\sum_{i=1}^{n} \beta_i {\tau_i}^{D-1} \log(\beta_i)}{\sum_{i=1}^{n} \beta_i {\tau_i}^{D-1} \log(\tau_i)}) = D.$$

Since $\dim \mu_{\mathfrak{p}} = \inf\{\dim_H A | \mu_{\mathfrak{p}}(A) = 1\}$ and $\mu_{\mathfrak{p}}(\Lambda_{\mathfrak{i}}) = 1$ we have

$$\dim_M \Lambda_{\mathfrak{i}} \geq \dim_H \Lambda_{\mathfrak{i}} \geq \dim \mu_{\mathfrak{p}} = D.$$

On the other hand by proposition 4.1 $\dim_M \Lambda_{\mathfrak{i}} \leq D$. □

6 Applying the theory of self-similar measures

Recall from section two that for a vector $\mathfrak{d} = (d_1, \dots, d_n) \in \mathbb{R}^n$ of translations, a probability vector $\mathfrak{p} = (p_1, \dots, p_n)$ and a vector $\mathfrak{b} = (\beta_1, \dots, \beta_n) \in (0,1)^n$ of contractions the projected measures $\hat{\mu}_{\mathfrak{p}}(\mathfrak{b}, \mathfrak{d})$ are self similar with respect to the maps

$$\hat{f}_i(x) = \beta_i x + d_i.$$

Furthermore recall that $(0, \mathrm{trans}(\mathfrak{b}, \mathfrak{d}))$ is an interval of transversallity for these measures. We have studies this class of self-similar measures in [8] using transversallity techniques. Our main result on absolute continuity of this measures is:

Theorem 6.1 *Fix a vector of translation* $\mathfrak{d} = (d_1, \dots, d_n) \in \mathbb{R}^n$*, a probability vector* $\mathfrak{p} = (p_1, \dots, p_n)$ *and a vector* $\mathfrak{a} = (\alpha_1, \dots, \alpha_n) \in (0,1)^n$*. Then for almost all*

$$v \in ((\frac{p_1}{\alpha_1})^{p_1}(\frac{p_2}{\alpha_2})^{p_2}\dots(\frac{p_n}{\alpha_n})^{p_n}, \mathrm{trans}(\mathfrak{a},\mathfrak{d}))$$

the measure $\hat{\mu}_{\mathfrak{p}}(v\alpha, \mathfrak{d})$ *is absolutely continues.*

Using theorem 6.1 and theorem 2.1 we are now able to proof theorem 2.2.

Proof of theorem 2.2 Fix a vector $\mathfrak{t}$ of contractions in the second coordinate direction and vectors of translations $\mathfrak{d}$ and $\mathfrak{e}$. These are chosen from the parameter domain $\mathfrak{I}$. If the parameter vector $\mathfrak{i} = (\mathfrak{b}, \mathfrak{t}, \mathfrak{d}, \mathfrak{e})$ is in addition chosen form the transversallity domain, as assumed in our theorem, we have $\mathfrak{b} \in \mathfrak{B}(\mathfrak{d})$ which means

$$\mathfrak{b} = v\mathfrak{a} = (v\alpha_1, \dots, v\alpha_n)$$

with

$$v \in ((\sum_{1=1}^{n} \alpha_i)^{-1}, \mathrm{trans}(\mathfrak{a},\mathfrak{d})).$$

By the lower bound on v the equation

$$\sum_{i=1}^{n} \alpha_i v \tau_i^{D-1}$$

has a solution $D > 1$. Now consider the probability vector $\mathfrak{p} = (\alpha_1 v \tau_1^{D-1}, \dots, \alpha_i v \tau_i^{D-1})$. Since

$$0 > \sum_{i=1}^{n} \alpha_i v \tau_i^{D-1} \log \tau^{D-1}$$

we have

$$\log(v) \geq \sum_{i=1}^{n} \alpha_i v \tau_i^{D-1} \log \tau^{D-1} v$$

and taking the exponential

$$v > \prod_{i=1}^{n} (v\tau_i^{D-1})^{\alpha_i v \tau_i^{D-1}} = \prod_{i=1}^{n} (\frac{p_i}{\alpha_i})^{p_i}$$

with $p_i = \alpha_i v \tau_i^{D-1}$. Hence theorem 6.1 applies and we get absolute continuity of the measures $\hat{\mu}_{\mathfrak{p}}(\mathfrak{b}, \mathfrak{d})$ for almost all $\mathfrak{b} \in \mathfrak{B}(\mathfrak{d})$. By theorem 2.1 the weighted moran formula generically holds for the dimension of $\Lambda_{\mathfrak{i}}$. □

References

[1] L. Barreira, Y. Pesin and J. Schmeling, *Dimension and product sructure of hyperbolic measures*, Ann. Math. 149, 755-83, 1999.

[2] T. Bedford, *Crinkly curves, Markov partitons and box dimension in self-similar sets*, Ph.D. Thesis, University of Warwick, 1984.

[3] K. Falconer, Fractal Geometry - Mathematical Foundations and Applications, Wiley, New York, 1990.

[4] J.E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. 30, 271-280, 1981.

[5] A. Katok and B. Hasselblatt, Introduction to Modern theory of dynamical Systems, Cambridge University press, 1995.

[6] Ledrappier and L.-S. Young, *The metric entropy of diffeomorphism*, Ann. Math. 122, 509-574, 1985.

[7] J. Neunhäuserer, *Dimension theory of linear solenoid*, Fractals, vol. 15, no. 1, 63-72, 2007.

[8] J. Neunhäuserer, *A general result on absolute continuity of non-uniform self-similar measures on the real line* , to appear in Fractals, 2008.

[9] Ya. Pesin, Dimension Theory in Dynamical Systems - Contemplary Views and Applications, University of Chicago Press, 1997.

[10] Y. Peres and B. Solomyak, *Self-similar measures and intersection of Cantor sets*, Trans. Amer. Math. Soc 350, no. 10, 4065-4087, 1998.

[11] M. Pollicott and K. Simon, *The Hausdorff dimension of λ-expansions with deleted digits*. Trans. Am. Math. Soc. 347, No.3, 967-983, 1995.

[12] C. McMullen, *The Hausdorff dimension of general Sierpinski carpets*, Nagoya Math. J. 96, 1-9, 1984.

[13] P. Moran, *Additive functions of intervals and Hausdorff measure*, Proc. Cambridge Philos. Soc. 42, 15-23, 1946.

[14] J. Schmeling and S. Troubetzkoy, *Dimension and invertibility of hyperbolic endomorphisms with singularities*, Ergod. Theo. and Dyn. Sys. 18, 1257-1282, 1998.