A general result on absolute continuity of non uniform self-similar measures on the real line

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Abstract

We present a general result on absolute continuity of biased non-uniform self-similar measure on the real line, given by n different contraction and translation rates. The result holds generically in the sense of Lebesgue measure on a certain part of the parameter domain.
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1 Introduction

In the last decade there was great success in the study of self-similar measures on the real line, see [1] and the bibliography there in. First of all Solomyak [2] proved that almost all Bernoulli convolutions, which are uniform self-similar measure, are absolutely continues with respect to Lebesgue measure and have a density in L^2. Then Peres and Solomyak considerately simplified the proof of this result and generalized it to biased Bernoulli convolutions, see [3], [4] and see [5]. In [6] we proved absolute continuity for a class of non-uniform self-similar measures, which are constructed by two different contractions. Furthermore in [7] and [8] self-similar measures with uniform contractions but n different translation rates were consider. Sufficient condition on absolute continuity of these measure are proved. These results can be applied in the study λ-expansions with deleted digits, compare [9].
In these paper we bring techniques together and prove a general theorem on absolute continuity of biased non-uniform self-similar measure constructed by n contractions with n different translation rates on the real line, see theorem 2.1. We get a lower bound on generic absolute continuity of the measures which is sharp and a condition for the density of the measures to be L^2. The proof of our result relies strongly on the transverselity technique used in all papers mentioned above. These technique restricts the upper bound of the parameter domain where we are able to proof absolute continuity. Also we do not belief that these restriction is necessary we are not able to avoid transverselity at this time.
The rest of the paper is organized as follows. In the next section we introduce our notation and present results. In section three we discuss a few special cases and applications. The last section contains the proof of the main theorem.

We like to thank the referees for pointing out the the first part of our theorem may be obtained in another way by using a very recent result on self-affine attractors, see [10].

2 Notations and results

Choose a vector $B = (\beta_1, \dots, \beta_n) \in (0,1)^n$ of contraction rates and a vector $D = (d_1, \dots, d_2) \in \mathbb{R}^n$ of translation rates and define an iterated function systems (IFS) by the lineare contractions

$$T_i x = \beta_i x + d_i \qquad i = 1, \dots n$$

on the real line $\mathbb{R}$. We know [11] that there is a unique compact attractor $\Lambda_{B,D} \subseteq \mathbb{R}$ of the IFS satisfying

$$\Lambda_{B,D} = \bigcup_{i=1}^{n} T_i(\Lambda_{B,D}).$$

Now given a probability vector $P = (p_1, \dots, p_n)$ we know [11] that there is unique Borel probability measure $\mu_{B,D}^P$ on $\Lambda_{B,D}$ satisfying

$$\mu_{B,D}^P = \sum_{i=1}^{n} p_i(\mu_{B,D}^P \circ T_i^{-1}).$$

The measure is $\mu_{B,D}^P$ is called self-similar with respect to the IFS. A more concrete realization of this measure is given by a nonlinear projection $\pi_{B,D}$ of the Bernoulli measure b^P on the sequence space $\Sigma = D^{\mathbb{N}_0}$ onto the real line. The projection map is given by

$$\pi_{B,D}(s) = \sum_{k=0}^{\infty} s_k \prod_{i=1}^{n} \beta_i^{\sharp_i^k(s)}$$

where

$$\sharp_i^k(s) = Card\{s_j | s_j = d_i \text{ for } j = 0, \dots, k-1\}.$$

for $s = (s_k) \in \Sigma$. With these notations we have:

Lemma 2.1

$$\mu_{B,D}^P = b^P \circ \pi_{B,D}^{-1}$$

Proof. A simple calculation shows that the projection $b^P \circ \pi_{B,D}^{-1}$ is self-similar with respect to the IFS. The result follows from uniqueness of self-similar measures. □

Self-similar measures are in general either totally singular or absolutely continuous. This is obvious by composing the measure into singular and absolute continuous part and using the uniqueness of a self-similarity given the IFS. There is one result on singularity of our measure using a dimension estimate, which seems to be nowadays folklore (compare books on dimension theory [12] or [13].

Proposition 2.1 *We have*

$$\dim \mu_{B,D}^P \leq \frac{\sum_{i=1}^{n} p_i \log(p_i)}{\sum_{i=1}^{n} p_i \log(\beta_i)}$$

Hence $\mu_{B,D}^P$ is singular if

$$\prod_{i=1}^{n} p_i^{p_i} > \prod_{i=1}^{n} \beta_i^{p_i}.$$

Sketch of proof. If we consider the metric on $D^{\mathbb{N}^0}$ given by

$$d(s,t) = \prod_{i=1}^{n} \beta_i^{\sharp_i^{\mathfrak{k}}(s)}$$

with $\mathfrak{k} = \min\{k | s_k \neq t_k\}$, than the Hausdorff dimension of the Bernoulli measures b^P on $(D^{\mathbb{N}^0}, d)$ is given by the righthand side of the dimension estimate. One proves this using the Shannon local entropy theorem for the numerator and Birkhoff's ergodic theorem to get denominater. Now the map $\pi_{B,D}^P$ is Lipschitz with respect to the metric d and does hence no increase dimension. □

In order to state our result on absolute continuity of self-similar measure we need the notion of transverselity. For $b > 0$ consider the space of analytic functions with $f(0) = 1$ and coefficients in the interval $[-b, b]$, i.e.

$$\mathcal{F}_b = \{f(x) = 1 + \sum_{k=1}^{\infty} b_k x^k | b_k \in [-b, b]\}$$

and let

$$t(b) = \min\{x > 0 | \exists f \in \mathcal{F}_b \text{ with } f(x) = f'(x) = 0\}.$$

Given an arbitrary $\epsilon > 0$ there is $\rho > 0$ such that each function $f \in \mathcal{F}_b$ crosses the x-axis transversely with slope in $[-\rho, \rho]$ on the interval of transverselity $[0, t(b) - \epsilon]$. This will be crucial in our proof of the absolute continuity of self similar measures. We like to mention the following result on the function t, see [14] and [15].

Proposition 2.2 *t is continuous and decreasing with*

$$t(1) = 0.64913... \text{ and } t(2) = 0,5 \text{ and } t(3) = 0.42772...$$

$$t(b) \geq (\sqrt{b} + 1)^{-1} \text{ for } b \in [1, 3 + \sqrt{8}]$$

$$t(b) = (\sqrt{b} + 1)^{-1} \text{ for } b \in [3 + \sqrt{8}, \infty).$$

In our context we let

$$b = \frac{\max\{\alpha_i d_j | d_j > 0\} + \max\{-\alpha_i d_j | d_j < 0\}}{\min_{i \neq j} |d_i - d_j|}$$

for an arbitrary vector $A = (\alpha_1, \dots, \alpha_n) \in (0, 1]^n$. Moreover choose an open interval $\mathfrak{I}(D, A)$ of ρ-transverselity in $[0, t(b)]$. With these notations we are prepared to satate our result

Theorem 2.1 *Fix a vector of translation $D = (d_1, \dots, d_n) \in \mathbb{R}^n$, a probability vector $P = (p_1, \dots, p_n)$ and a vector $A = (\alpha_1, \dots, \alpha_n) \in (0, 1]^n$. For almost all*

$$\beta \in ((\frac{p_1}{\alpha_1})^{p_1} (\frac{p_2}{\alpha_2})^{p_2} \dots (\frac{p_n}{\alpha_n})^{p_n}, 1) \cap \mathfrak{I}(D, A)$$

the self-similar measure $\mu_{\beta A,D}^P$ with contraction vector $(\beta\alpha_1, \dots, \beta\alpha_n)$ is absolutely continuous with respect to the Lebesgue measure and has a density in L^2 for almost all

$$\beta \in (\frac{p_1^2}{\alpha_1} + \frac{p_2^2}{\alpha_2} + \dots + \frac{p_n^2}{\alpha_n}, 1) \cap \mathfrak{I}(D, A).$$

Using the theorem of Fubini our result obviously implies absolute continuity of the measures $\mu^P_{\beta A,D}$ for almost all contractions rates in the sense of n-dimensional Lebesgue measure on the appropriate parameter domain. Note that by proposition 2.1 the lower bound on absolute continuity given in our theorem is in fact sharp. We conjecture here that the lower bound for the density of the measures to be in L^2 is sharp as well.

3 A few Special cases and applications

Let us discuss a few special and applications of theorem 2.1.

1. Case Setting $D = \{0,1\}$, $P = (1/2, 1/2)$ and $A = (1,1)$ our result directly implies absolute continuity of almost all Bernoulli convolutions b_β for $\beta \in (0.5, 0.649)$. To get this result for the whole interval $(0,1)$ one has to use addition arguments using Fourier transforms, see [16].

2. Case With $D = (0,1)$, $P = (1/2, 1/2)$ and $A = (1,c)$ with $c \in (0,1)$ the general result implies absolutely continuity of non-uniform self-similar measures $b_{\beta,c\beta}$ for almost all $\beta \in (1/2\sqrt{c}, 0, 649)$ and density in L^2 for almost all $\beta \in (1/4 + 1/(4c), 0, 649)$. This is exactly theorem I of [17] for $q = 1, 2$.

3. Case Setting $A = (1, 1, \ldots 1)$ and choosing a probability vector $P = (p_1, \ldots, p_n)$ and a translation vector $D = (d_1, \ldots, d_n)$ we get from the L^2 part of theorem 2.1 a result very similar to theorem 4.3 of [18]. A difference is that the upper bound on the transverselity interval may be chosen slightingly larger in this special case. On the other hand our results extends theorem 4.3 of [18], since our lower bound on absolute continuity $\prod p_i^{p_i}$ is sharp.

4. Case Let $D \in (0, 2]^n$ and $P = (1/n, 1/n, \ldots, 1/n)$. We get generic absolute continuity of the corresponding measures in $(1/(n\sqrt[n]{\alpha_1\alpha_2 \ldots \alpha_n}), 0.5)$ and density in L^2 in $(1/n^2 \sum \frac{1}{\alpha_i}, 0.5)$. As far as we know results of this type are completely new. To consider one example let $n = 3$ and $A = (1, 2/3, 3/4)$. For almost all $\beta \in (1/(3\sqrt[3]{2}), 0.5)$ the self-similar measure $\mu^P_{B,D}$ with $B = (\beta, 2\beta/3, 3\beta/4)$ is absolutely continuous and has a density in L^2 for almost $\beta \in (23/54, 1/2)$.

Of course one might consider other intersting special cases for given purposes.

We like to mention here a application of theorem 2.1 to the solution of certain functional equations. Consider the n-scale difference equation given by:

$$f(x) = \sum_{i=1}^{n} c_i f(a_i x - b_i).$$

By setting $f(x) = \mu^P_{\beta A,D}((\infty, x])$ we get L^1 resp. L^2 solutions of the following equations of this type

$$f(x) = \sum_{i=1}^{n} p_i f((\beta\alpha_i)^{-1}x - (d_i\beta\alpha_i)^{-1}).$$

under the assumptions of theorem 2.1.

Furthermore we like to mention an application of our result to the self-similar sets $\Lambda_{\beta A,D}$. If we have an absolutely continuous self-similar measure on a self-similar set than this set has obviously positive Lebesgue measure. Thus under the assumption of theorem 2.1 we have $\ell(\Lambda^P_{\beta A,D}) > 0$. It is in general a non-trivial problem to proof this directly.

There are also applications of our theorem to the dimension theory of self-affine sets. We will discuss this in another paper.

4 Proof of the result

Fix A, D and P as in the assumption of theorem 2.1.
We first prove the result for density in L^2, than generalize it to density in L^q for $q \in (1,2]$ and consider the limit.
We use the differentation method of Mattila [19] to prove absolute continuity of the measures $\mu^P_{\beta A,D}$. Consider the lower derivative of these measure defined by

$$\underline{\mu^P_{\beta A,D}}'(x) := \liminf_{r \longmapsto 0} \frac{\mu^P_{\beta A,D}(B_r(x))}{2r}.$$

With the help of Vitali‘s covering theorem one proofs that $\mu^P_{\beta A,D}$ is absolutely continuous with respect to the Lebesgue measure if and only if $\underline{\mu^P_{\beta A,D}}'(x) < \infty$ for almost $x \in \mathbb{R}$, see [19]. Hence if we show

$$\mathfrak{E} := \int_{\mathfrak{I}} \int_{\mathbb{R}} \underline{\mu^P_{\beta A,D}}'(x) d\mu^P_{\beta A,D}(x) d\beta < \infty$$

for $\mathfrak{I} := (\beta_0, 1) \cap \mathfrak{I}(D, A)$ with $\beta_0 > p_1^2/\alpha_1 + p_2^2/\alpha_2 + \ldots + p_n^2/\alpha_n$ arbitrary, we have proved generic absolute continuity of the measures $\mu^P_{\beta A,D}$. Moreover if a measure $\mu^P_{\beta A,D}$ is absolutely continues for some β then $\underline{\mu^P_{\beta A,D}}'$ is the Radon-Nykodien derivative $d\mu^P_{\beta A,D}/dx$, so

$$\mathfrak{E} := \int_{\mathfrak{I}} \int_{\mathbb{R}} (d\mu^P_{\beta A,D}/dx)^2 dx d\beta < \infty$$

proving that the density of the measures is generically in L^2.

Now we begin to estimate the integral. By Fatous lemma we have

$$\mathfrak{E} \leq \liminf_{r \longmapsto 0} (2r)^{-1} \int_{\mathfrak{I}} \int_{\mathbb{R}} \mu^P_{\beta A,D}(B_r(x)) d\mu^P_{\beta A,D}(x) d\beta.$$

Now we change variables using $\mu^P_{B,D} = b^P \circ \pi^{-1}_{B,D}$ and get

$$\mathfrak{E} \leq \liminf_{r \longmapsto 0} (2r)^{-1} \int_{\mathfrak{I}} \int_{D^{\mathbb{N}_0}} \mu^P_{\beta A,D}(B_r(\pi_{\beta A,D}(s))) db^P(s) d\beta.$$

Let $\mathbf{1_M}$ be the characteristic function of a set $\mathbf{M}$. Using $\mu^P_{B,D} = b^P \circ \pi^{-1}_{B,D}$ again we get

$$\mu^P_{\beta A,D}(B_r(\pi_{\beta A,D}(s))) = \int_{\mathbb{R}} \mathbf{1}_{\mathbf{B_r}(\pi_{\beta \mathbf{A},\mathbf{D}}(\mathbf{s}))}(x) d\mu^P_{\beta A,D}(x)$$

$$= \int_{D^{\mathbb{N}_0}} \mathbf{1}_{\{\mathbf{t} \in \mathbf{D}^{\mathbb{N}_0} || \pi_{\beta \mathbf{A}, \mathbf{D}}(\mathbf{t}) - \pi_{\beta \mathbf{A}, \mathbf{D}}(\mathbf{s})| \leq \mathbf{r}\}} db^P(t)$$

Hence

$$\mathfrak{E} \leq \liminf_{r \longmapsto 0} (2r)^{-1} \int_{\mathfrak{I}} \int_{D^{\mathbb{N}_0}} \int_{D^{\mathbb{N}_0}} \mathbf{1}_{\{\mathbf{t} \in \mathbf{D}^{\mathbb{N}_0} || \pi_{\beta \mathbf{A}, \mathbf{D}}(\mathbf{t}) - \pi_{\beta \mathbf{A}, \mathbf{D}}(\mathbf{s})| \leq \mathbf{r}\}} db^P(t) db^P(s) d\beta$$

$$= \liminf_{r \longmapsto 0} (2r)^{-1} \int_{D^{\mathbb{N}_0}} \int_{D^{\mathbb{N}_0}} \int_{\mathfrak{I}} \mathbf{1}_{\{\mathbf{t} \in \mathbf{D}^{\mathbb{N}_0} || \pi_{\beta \mathbf{A}, \mathbf{D}}(\mathbf{t}) - \pi_{\beta \mathbf{A}, \mathbf{D}}(\mathbf{s})| \leq \mathbf{r}\}} d\beta db^P(t) db^P(s)$$

$$= \liminf_{r \longmapsto 0} (2r)^{-1} \int_{D^{\mathbb{N}_0}} \int_{D^{\mathbb{N}_0}} \ell(\{\beta \in \mathfrak{I} || \pi_{\beta A, D}(t) - \pi_{\beta A, D}(s)| \leq r\}) db^P(t) db^P(s)$$

by changing the order of integration and integrating. Here ℓ demotes the one dimensional Lebesgue measure.
We now have to use transverselity to estimate the integrant. To this end let

$$\phi_{s,t}(\beta) = \pi_{\beta A, D}(t) - \pi_{\beta A, D}(s) = \sum_{k=0}^{\infty} (t_k \prod_{i=1}^{n} (\beta \alpha_i)^{\sharp_i^k(t)} - s_k \prod_{i=1}^{n} (\beta \alpha_i)^{\sharp_i^k(s)})$$

$$= \sum_{k=0}^{\infty} (t_k \prod_{i=1}^{n} \alpha_i^{\sharp_i^k(t)} - s_k \prod_{i=1}^{n} \alpha_i^{\sharp_i^k(s)}) \beta^k$$

$$= \beta^{\mathfrak{k}} \sum_{k=0}^{\infty} (t_{\mathfrak{k}+k} \prod_{i=1}^{n} \alpha_i^{\sharp_i^{\mathfrak{k}+k}(t)} - s_{k+\mathfrak{k}} \prod_{i=1}^{n} \alpha_i^{\sharp_i^{\mathfrak{k}+k}(s)}) \beta^k$$

$$= \beta^{\mathfrak{k}} (t_{\mathfrak{k}} \prod_{i=1}^{n} \alpha_i^{\sharp_i^{\mathfrak{k}}(t)} - s_{\mathfrak{k}} \prod_{i=1}^{n} \alpha_i^{\sharp_i^{\mathfrak{k}}(s)}) (1 + \sum_{k=1}^{\infty} \frac{(t_{\mathfrak{k}+k} \prod_{i=1}^{n} \alpha_i^{\sharp_i^{\mathfrak{k}+k}(t)} - s_{\mathfrak{k}+k} \prod_{i=1}^{n} \alpha_i^{\sharp_i^{\mathfrak{k}+k}(s)})}{(t_{\mathfrak{k}} \prod_{i=1}^{n} \alpha_i^{\sharp_i^{\mathfrak{k}}(t)} - s_{\mathfrak{k}} \prod_{i=1}^{n} \alpha_i^{\sharp_i^{\mathfrak{k}}(s)})} \beta^k)$$

$$= \beta^{\mathfrak{k}} (t_{\mathfrak{k}} \prod_{i=1}^{n} (\alpha_i)^{\sharp_i^{\mathfrak{k}}(t)} - s_{\mathfrak{k}} \prod_{i=1}^{n} (\alpha_i)^{\sharp_i^{\mathfrak{k}}(s)}) g_{s,t}(\beta)$$

$$= \beta^{\mathfrak{k}} (t_{\mathfrak{k}} - s_{\mathfrak{k}}) \prod_{i=1}^{n} (\alpha_i)^{\sharp_i^{\mathfrak{k}}(t)} g_{s,t}(\beta)$$

where $\mathfrak{k} = \min\{k | s_k \neq t_k\}$ and

$$g_{s,t}(\beta) = 1 + \sum_{k=1}^{\infty} c_k(s,t) \beta^k.$$

In order to establish ρ-transverselity of the power series g we have to estimate

$$|c_k(s,t)| = |\frac{(t_{\mathfrak{k}+k} \prod_{i=1}^{n} \alpha_i^{\sharp_i^{\mathfrak{k}+k}(t)} - s_{\mathfrak{k}+k} \prod_{i=1}^{n} \alpha_i^{\sharp_i^{\mathfrak{k}+k}(s)})}{(t_{\mathfrak{k}} \prod_{i=1}^{n} \alpha_i^{\sharp_i^{\mathfrak{k}}(t)} - s_{\mathfrak{k}} \prod_{i=1}^{n} \alpha_i^{\sharp_i^{\mathfrak{k}}(s)})}|$$

$$= |\frac{(t_{\mathfrak{k}+k} \prod_{i=1}^{n} \alpha_i^{\sharp_i^{k}(\sigma^{\mathfrak{k}}(t))} - s_{\mathfrak{k}+k} \prod_{i=1}^{n} \alpha_i^{\sharp_i^{k}(\sigma^{\mathfrak{k}}(s))})}{(t_{\mathfrak{k}} - s_{\mathfrak{k}})}|$$

$$\leq \frac{\max\{\alpha_i d_j | d_j > 0\} + \max\{-\alpha_i d_j | d_j < 0\}}{\min_{i\neq j} |d_i - d_j|}.$$

By assumption the function g satisfies ρ-transverselity for $\beta \in \mathfrak{I}$, hence we can estimate integrand by:

$$\ell(\{\beta \in \mathfrak{I} || \pi_{\beta A,D}(t) - \pi_{\beta A,D}(s)| \leq r\})$$

$$= \ell(\{\beta \in \mathfrak{I} || g_{s,t}(\beta)| \leq r\beta^{-\mathfrak{k}}(t_{\mathfrak{k}} - s_{\mathfrak{k}})^{-1}(\prod_{i=1}^{n} \alpha_i^{-\sharp_i^{\mathfrak{k}}(t)}\}$$

$$\leq \ell(\{\beta \in \mathfrak{I} || g_{s,t}(\beta)| \leq r\beta_0^{-\mathfrak{k}}(t_{\mathfrak{k}} - s_{\mathfrak{k}})^{-1} \prod_{i=1}^{n} \alpha_i^{-\sharp_i^{\mathfrak{k}}(t)}\}$$

$$\leq 2C\rho^{-1} r \beta_0^{-\mathfrak{k}} \prod_{i=1}^{n} \alpha_i^{-\sharp_i^{\mathfrak{k}}(t)}$$

where $C = \max\{|d_u - d_v|^{-1} | 1 \leq u, v \leq n\}$. Now pasting this estimate into the last integral estimate on $\mathfrak{E}$ we get

$$\mathfrak{E} \leq C\rho^{-1} \int_{D^{\mathbb{N}_0}} \int_{D^{\mathbb{N}_0}} \beta_0^{-\mathfrak{k}} \prod_{i=1}^{n} \alpha_i^{-\sharp_i^{\mathfrak{k}}(t)} db^P(t) db^P(s).$$

With the notation $\mathfrak{k} = \min\{k | s_k \neq t_k\}$ we have been suppressing indices which we now need. So write $\mathfrak{k} = |s \wedge t|$. With this we will calculate the integral

$$\int_{D^{\mathbb{N}_0}} \int_{D^{\mathbb{N}_0}} \beta_0^{-|s\wedge t|} \prod_{i=1}^{n} \alpha_i^{-\sharp_i^{|s\wedge t|}(t)} db^P(t) db^P(s)$$

$$= \sum_{k=0}^{\infty} \beta_0^{-k} \sum_{k_1+k_2+\ldots k_n = k} (\prod_{i=1}^{n} \alpha_i^{-k_i}) b^P \times b^P\{(s,t) \in (D^{\mathbb{N}_0})^2 || s \wedge t| = k \quad \sharp_i^k(s) = \sharp_i^k(t) = k_i\}$$

$$= \sum_{k=0}^{\infty} \beta_0^{-k} \sum_{k_1+k_2+\ldots k_n = k} (\prod_{i=1}^{n} \alpha_i^{-k_i}) \frac{k!}{k_1! k_2! \ldots k_n!} (\prod_{i=1}^{n} p_i^{2k_i})$$

$$= \sum_{k=0}^{\infty} \beta_0^{-k} (\frac{p_1^2}{\alpha_1} + \ldots + \frac{p_n^2}{\alpha_n})^k$$

Now $\beta_0 > p_1^2/\alpha_1 + p_2^2/\alpha_2 + \ldots + p_n^2/\alpha_n$ hence the geometric series converges and $\mathfrak{E} < \infty$ concluding the proof of absolutely continuity with density in L^2.

Now fix $q \in (1, 2]$. As in the case $q = 2$, we generically get a density in L^q for the measures $\mu_{\beta A,D}^P$ if we show

$$\mathfrak{E}_q := \int_{\mathfrak{I}_q} \int_{\mathbb{R}} (\underline{\mu_{\beta A,D}^P}')^{q-1}(x) d\mu_{\beta A,D}^P(x) d\beta < \infty$$

where the domain $\mathfrak{I}_q \subseteq \mathfrak{I}(D,A)$ will be specified below. Using Hölder inequality for integrals we see that

$$\mathfrak{E}_\mathfrak{q} \leq \liminf_{r\longmapsto 0}(2r)^{1-q}\int_{\mathfrak{I}_\mathfrak{q}}(\int_{\mathbb{R}}\mu^P_{\beta A,D}(B_r(x))d\mu^P_{\beta A,D}(x))^{q-1}d\beta.$$

Now using exactly the same line of reasoning as above the integral is up to a constant bounded by

$$\int_{D^{\mathbb{N}_0}}(\int_{D^{\mathbb{N}_0}}\beta_0^{-|s\wedge t|}\prod_{i=1}^{n}\alpha_i^{-\sharp_i^{|s\wedge t|}(t)}db^P(t))^{q-1}db^P(s).$$

Now using Hölder inequality the expression is bounded by

$$\sum_{k=0}^{\infty}\beta_0^{-k(q-1)}\sum_{k_1+k_2+\ldots k_n=k}(\prod_{i=1}^{n}\alpha_i^{-k_i(q-1)})\frac{k!}{k_1!k_2!\ldots k_n!}(\prod_{i=1}^{n}p_i^{qk_i})$$

$$=\sum_{k=0}^{\infty}\beta_0^{-k(q-1)}(\frac{p_1^q}{\alpha_1^{q-1}}+\ldots+\frac{p_n^q}{\alpha_n^{q-1}})^k.$$

Hence the measure $\mu^P_{\beta A,D}$ is absolutely continues with density in L^q for almost all

$$\beta \in \mathfrak{I}_\mathfrak{q} := ((\frac{p_1^q}{\alpha_1^{q-1}}+\frac{p_2^q}{\alpha_2^{q-1}}+\ldots+\frac{p_n^q}{\alpha_n^{q-1}})^{\frac{1}{q-1}},1)\cap\mathfrak{I}(D,A).$$

Taking $q \longmapsto 1$ and using the L'Hospital's rule this gives the lower bound on absolute continuity stated in our theorem. □

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