A construction of singular overlapping asymmetric self-similar measures

J. Neunhäuserer

Reitstallweg 9,
38640 Goslar, Germany
neunchen@aol.com

Abstract

In [8] we found a class of overlapping asymmetric self-similar measures on the real line, which are generically absolutely continuous with respect to the Lebesgue measure. Here we construct exceptional measures in this class being singular.
MSC 2000: Primary 28A12, Secondary 28A78, 28A75, 14H50

1 Introduction

In the last century Bernoulli convolutions on the real line, which may also be described as symmetric self similar measures, were successfully studied, see [12]. Generically these measures are absolutely continuous with quare integrable density and they get singular for certain algebraic parameters, see [2] [3], [15] and [11]. In [8] we began to study overlapping asymmetric self-similar measures on the reale line, which generalize Bernoulli convolutions. The parameter domain of these measures is two dimensional. We extended arguments of Peres and Solomyak [11] to show that overlapping asymmetric self-similar measures are generically (in the sense of Lebesgue measure on the parameter domain) absolutely continuous with respect to the Lebesgue measure on the real line (see theorem 2.1). In this article we are concerned with the question, if there exists any exceptional overlapping asymmetric self-similar measures being singular. Looking at the symmetric case this is quiet reasonable. We will prove here that near to the boundary of the parameter domain there exists exceptional values for which overlapping asymmetric self-similar measures have a dimension drop and get hence singular (see theorem 3.2 below). In fact we will show even more, each point of this boundary is an accumulation point of parameters for which the dimension of the measures is less than one. Our strategy to get this result is to proof a dimension estimate which is effective near to the boundary of the parameter domain (see theorem 3.1 below). We proof this dimension estimate using a meanwhile classical approach in ergodic theory. The dimension of our measures is bounded by the quotient of entropy and Lyapunov exponent and we are able to estimate the dimension and calculate the Lyapunov exponent, compare [13] and especially [10]. The assumption of our dimension estimate is that the parameters values, of the measures we consider, fulfill a certain type of algebraic equation in two variables. We have thus to guarantee the existence of such solutions in an appropriate domain (see proposition 3.1). To proof this we use a refinement of construction invented by Simon and Solomyak in the context of self-similar sets, see [14].

The rest of this paper is organized as follows. In section two we introduce our notations and basic objects, in section three we present our results, in section four we proof the dimension estimate on asymmetric self-similar measures and in section five we proof the existence of solutions of a certain type of algebraic equations we used.

2 Preliminaries

For $\beta_1, \beta_2 \in (0,1)$ consider the linear contractions

$$T_1 x = \beta_1 x \qquad T_2 x = \beta_2 x + 1$$

on the interval $I = [0, 1/(1-\beta_2)]$. For a finite sequence $s = (s_1, s_2, \ldots, s_n) \in \Sigma_n := \{1,2\}^n$ we let

$$T_s = T_{s_1} \circ T_{s_2} \circ \ldots \circ T_{s_n}$$

Now consider infinite sequences $s \in \Sigma := \{1,2\}^{\mathbb{N}}$. We define a map $\pi : \Sigma \longmapsto I$ by

$$\pi(s) = \lim_{n \longmapsto \infty} T_{s_1} \circ T_{s_2} \circ \ldots \circ T_{s_n}(I).$$

For $s \in \Sigma_n$ or $s \in \Sigma$ and $k \in \mathbb{N}$ let

$$1_k(s) = Card\{i | s_i = 1 \text{ for } i = 1 \ldots k\}$$

$$2_k(s) = Card\{i | s_i = 2 \text{ for } i = 1 \ldots k\}$$

By induction we see that for $s \in \Sigma_n$ we have

$$T_s x = \beta_1^{1_n(s)} \beta_2^{2_n(s)} x + V_s \qquad (\star)$$

where

$$V_s = \sum_{k=1}^{n} (s_k - 1) \beta_1^{1_{k-1}(s)} \beta_2^{2_{k-1}(s)}.$$

Accordingly the map π has the following explicit form,

$$\pi(s) = \sum_{k=1}^{\infty} (s_k - 1) \beta_1^{1_{k-1}(s)} \beta_2^{2_{k-1}(s)}.$$

It is well known, see [4] or [1], that for all $\beta_1, \beta_2 \in (0,1)$ there exists a unique Borel probability measure $\mu = \mu_{\beta_1,\beta_2}$ on I with the following self-similarity,

$$\mu = \frac{T_1(\mu) + T_2(\mu)}{2}.$$

If b is the equally weighted Bernoulli measure on Σ we get the following description of the equally weighted self-similar measures μ using the coding map π,

$$\mu = \pi(b) = b \circ \pi^{-1}.$$

If $\beta_1 = \beta_2 = \beta \in (0,1)$ the measures $\mu = \mu_{\beta,\beta}$ are infinite Bernoulli convolutions. As we mentioned in the introduction these measures were successfully studied. In the case $\beta_2 \neq \beta_1$ we call the measures $\mu = \mu_{\beta_1,\beta_2}$ **asymmetric self-similar**. We have the following result on the properties of these measures:

Theorem 2.1 *If $\beta_1, \beta_2 \in (0,1)$ and $\beta_1\beta_2 < 1/4$ than the measure μ_{β_1,β_2} is singular with*

$$\dim_H \mu_{\beta_1,\beta_2} < -2 \log 2 / (\log \beta_1 + \log \beta_2)$$

For almost all $\beta_1, \beta_2 \in (0, 0.649)$ with $\beta_1\beta_2 \geq 1/4$ the measures μ_{β_1,β_2} are absolutely continuous.

This theorem is an immediate consequence of Theorem I in [8]. There we worked in a more general setting also considering measures wich are not equally weighted and studying the density of these measures. We like to mention that Ngai and Wang have related results, see [7].
If $\beta_1\beta_2 \geq 1/4$ we call μ_{β_1,β_2} **overlapping self-similar**. We do not belief that the bound 0.649 in our result on overlapping self-similar measures is essential, also for some technical reasons we are not able to remove it. In view of our theorem it is a natural question if there are any singular overlapping asymmetric self-similar measures μ_{β_1,β_2} at all. Let us first remark that in the symmetric situation $\beta = \beta_1 = \beta_2$ the measure $\mu_{\beta,\beta}$ gets singular, if $\beta \in (0.5, 1)$ is the reciprocal of a Pisot number[1]. Thus one may think that there should be some algebraic equations for $\beta_1, \beta_2 \in (0, 1)$ with $\beta_1\beta_2 > 1/4$ and $\beta_1 \neq \beta_2$ such that μ_{β_1,β_2} gets singular. We show in this article, that this is in fact true.

3 Results

We first present here a new dimension estimate on asymmetric self-similar measures. By $\dim_H \mu$ we denote the Hausdorff dimension of a measure μ given by

$$\dim_H \mu = \inf\{\dim_H M | \mu(M) = 1\}\}$$

where $\dim_H M$ is the Hausdorff dimension of a set. See [1] or [13] for a good introduction to dimension theory. Now we state our result.

Theorem 3.1 *Let $\beta_1, \beta_2 \in (0, 1)$ and $s, t \in \Sigma_n$ with $s \neq t$. If $T_s = T_t$ then*

$$\dim_H \mu_{\beta_1,\beta_2} < -\frac{2\log(2^n - 1)}{n(\log \beta_1 + \log \beta_2)}.$$

This dimension estimate has the following obvious corollary on the singularity of asymmetric self-similar measures.

Corollary 3.1 *Let $\beta_1, \beta_2 \in (0, 1)$ and $s, t \in \Sigma_n$ with $s \neq t$. If $T_s = T_t$ and*

$$\frac{1}{(\sqrt[n]{2^n - 1})^2} > \beta_1\beta_2$$

then μ_{β_1,β_2} is singular with $\dim_H \mu_{\beta_1,\beta_2} < 1$.

We remain here with the problem if there are any (β_1, β_2) in the domain

$$D_n = \{(\beta_1, \beta_2) \in (0, 1) | \frac{1}{(\sqrt[n]{2^n - 1})^2} > \beta_1\beta_2 > \frac{1}{4}\}$$

such that $T_s = T_t$ for $s, t \in \Sigma_n$ with $s \neq t$. By equation ($\star$) on the last side this is equivalent to find solutions of an algebraic equations $V_s = V_t$ with $s \neq t$ and $1_n(s) = 1_n(t)$ in the domain D_n. We will adopt a construction developed by Simon and Solomayak [14] to prove the following proposition.

[1]An algebraic integer with conjugates inside the unit circle, see [2]

Proposition 3.1 *For all* $\beta_2 \in (1/4, 1/2)$ *there exists constants* $c > 0$, $\lambda \in (0,1)$, *a sequence* $n_k \longmapsto \infty$, *an*

$$\beta_1 \in (\frac{1}{4\beta_2}, \frac{1}{4\beta_2} + c\lambda^{n_k})$$

and $s, t \in \Sigma_{n_k}$ *with* $s \neq t$ *such that* $T_s = T_t$.

By this proposition and corollary 3.1 we get the main result of the articel

Theorem 3.2 *For all* $\beta_2 \in (1/4, 1/2)$ *and* $\epsilon > 0$ *sufficient small there is an*

$$\beta_1 \in (\frac{1}{4\beta_2}, \frac{1}{4\beta_2} + \epsilon)$$

such that the measure μ_{β_1,β_2} *is singular with* $\dim_H \mu_{\beta_1,\beta_2} < 1$.

Proof. Fix $\beta_2 \in (1/4, 1/2)$ and c, λ from proposition 3.1. By the rule of Le Hospital the convergence of $(\sqrt[n]{2^n - 1})^2$ to 4 is subexponential hence there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 : \frac{1}{4} + c\beta_2\lambda^n < \frac{1}{(\sqrt[n]{2^n - 1})^2}$$

Choose in proposition 3.1 an $n_k > n_0$ and fix $\epsilon_0 = c\lambda^{n_k}$. Now by proposition 3.1 for $\epsilon < \epsilon_0$ there is an $n_k > n_0$ with $c\lambda^{n_k} < \epsilon$ and there exists

$$\beta_1 \in (\frac{1}{4\beta_2}, \frac{1}{4\beta_2} + \epsilon)$$

with $T_s = T_t$ for $s, t \in \Sigma_{n_k}$. In addition we have

$$1/4 < \beta_1\beta_2 < 1/4 + \beta_2 c\lambda^{n_k} < \frac{1}{(\sqrt[n_k]{2^{n_k} - 1})^2}$$

Hence $(\beta_1, \beta_2) \in D_{n_k}$ and by corollary 3.1 we get $\dim_H \mu_{\beta_1,\beta_2} < 1$ implying singularity of the measure μ_{β_1,β_2} . □

We may paraphrase our result as follows. In the parameter domain where overlapping asymmetric self similar measures are generically absolutely continues we find exceptional parameter values, sufficient near to any point of the boundary, for which these measures are singular. Each point of this boundary is an accumulation point of such exceptional values. Of course our result leaves it open if there are any exceptional values near to points with a given distance from the boundary. Our dimension estimate in theorem 3.1 is not strong enough to answer this question. We conjecture from an analogy with Bernoulli convolutions that this estimate is not sharp in general (compare with [9]).

4 A dimension estimate for asymmetric self-similar measures

Our aim in this section is to proof theorem 3.1 using a classical dimension estimate by entropy and Lyapunov exponent. To this end we first introduce the Lyapunov exponent $\Xi(\mu)$ of the self-similar measure $\mu = \mu_{\beta_1,\beta_2}$. For $s \in \Sigma$ let

$$\Xi_s(\mu) = \lim_{n \longmapsto \infty} \frac{1}{n} \log ||T_{s_1} \circ T_{s_2} \circ \ldots \circ T_{s_n}|| = \lim_{n \longmapsto \infty} \frac{1}{n} \log \beta_1^{1_n(s)} \beta_2^{2_n(s)}$$

The following lemma is just an application of Birkhoff‘s ergodic theorem, see for instance [5].

Lemma 4.1 *For allmost all $s \in \Sigma$ with respect to b we have*

$$\Xi_s(\mu) = -\frac{1}{2}(\log\beta_1 + \log\beta_2) =: \Xi(\mu)$$

Proof. Just apply the ergodic theorem to the function $f : \Sigma \longmapsto \mathbb{R}$

$$f(s) = \{ \begin{array}{lll} \log\beta_1 & \text{for} & s_1 = 1 \\ \log\beta_2 & \text{for} & s_1 = 2 \end{array}$$

□

Now we introduce the entropy $h(\mu)$ of the self-similar measure μ. For $m \in \mathbb{N}$ define a Partition $\mathbf{P}_n$ of Σ by the following equivalence relation

$$s \sim t :\Leftrightarrow T_{s_1} \circ T_{s_2} \circ \ldots \circ T_{s_m} = T_{t_1} \circ T_{t_2} \circ \ldots \circ T_{t_m}.$$

The entropy of the partition is given by

$$H(\mathbf{P}_m) = -\sum_{P \in \mathbf{P}_n} b(P) \log b(P).$$

The sequence $H(\mathbf{P}_n)$ is subadditive hence the limit

$$h(\mu) := \lim_{m \longmapsto \infty} \frac{H(\mathbf{P}_m)}{m}$$

exists. In the following lemma we state a simple estimate on the entropy $h(\mu)$.

Lemma 4.2 *If $T_s = T_t$ holds for $s, t \in \Sigma_n$ with $s \neq t$, then we have*

$$h(\mu) \leq \frac{1}{n}\log(2^n - 1).$$

Proof. It is well knowen that

$$H(\mathbf{P}_m) \leq \log(Card(\mathbf{P}_m))$$

see for instance [5]. By the definition of $\mathbf{P}_m$ and our assumption we have

$$Card(\mathbf{P}_{nm}) \leq (2^n - 1)^m$$

Hence

$$h(\mu) = \lim_{m \longmapsto \infty} \frac{H(\mathbf{P}_{nm})}{nm} \leq \lim_{m \longmapsto \infty} \frac{\log(Card(\mathbf{P}_{nm}))}{nm} = \frac{1}{n}\log(2^n - 1)$$

□

In the dimensiontheory it has be shown that for a broad classes of measures the Hausdorff dimension is bounded from above by the quotient of entropy and Lyapunov exponent, see [13]. For symmetric self-similar measures this result is contained in [6] and for general classes measures constructed by iterated function systems it is proofed in [10]. The following proposition is just an application of theorem 2.2 in[10] to the iterated function system (T_1, T_2).

Proposition 4.1 *With the notations from above*

$$\dim_H \mu \leq \frac{h(\mu)}{\Xi(\mu)}$$

Theorem 3.1 is now obvious from the results of this section.

5 Existence of solutions of certain algebraic equations in two variables

In this section we will proof proposition 3.1. We have to find sequences $s, t \in \Sigma_n$ with $s \neq t$ such that $T_s = T_t$ holds for some $\beta_1, \beta_2 \in (0,1)$. We will use here a slight modification of a construction we found in the proof of proposition 3.4. of Simon and Solomyaks work on self-similar sets [14].
Let $s \in \Sigma_n$ be the sequence given by $12^{N_1}1^{M_1} \ldots 1^{M_{k-1}}2^{N_k}1^{M_k}$ where we use the exponential here to describe the number of repetitions of an entry. Let $\mathcal{N} = \mathcal{N}_k = N_1 + \ldots N_k$, $\mathcal{M} = \mathcal{M}_k = 1 + M_1 + \ldots + M_k$ and $n = n_k = \mathcal{M}_k + \mathcal{N}_k$. Furthermore let $t \in \Sigma_n$ be given by $21^{\mathcal{M}+1}2^{\mathcal{N}-1}$. By the definition of T_s and T_t we get:

$$T_s x = \beta_1^{\mathcal{N}} \beta_2^{\mathcal{M}} x + V_s(\beta_1, \beta_2) \quad \text{and} \quad T_t x = \beta_1^{\mathcal{N}} \beta_2^{\mathcal{M}} x + V_t(\beta_1, \beta_2)$$

with

$$V_s(\beta_1, \beta_2) = \sum_{j=0}^{k-1} (\beta_2^{N_1 + \ldots + N_j} \beta_1^{1 + M_1 + \ldots + M_j} \sum_{i=0}^{N_{j+1}-1} \beta_2^i)$$

$$V_t(\beta_1, \beta_2) = 1 + \beta_1^{\mathcal{M}} \sum_{i=1}^{\mathcal{N}-1} \beta_2^i$$

Obviously we have $T_s = T_t$ if and only if the algebraic equation

$$V_s(\beta_1, \beta_2) = V_t(\beta_1, \beta_2)$$

holds. We first construct by recursion a sequences of natural numbers M_l and N_l such that $V_s(\beta_1, \beta_2)$ is arbitrary close to $V_t(\beta_1, \beta_2)$. Let

$$N_1 = \max\{N \geq 1 | y_1 := \beta_1 \sum_{i=0}^{N-1} \beta_2^i < 1\}$$

$$M_1 = \min\{M \geq 1 | y_1 + \beta_2^{N_1} \beta_1^{M+1} < 1\}$$

If N_{l-1} M_{l-1} an y_{l-1} are constructed continue with

$$N_l = \max\{N \geq 1 | y_l = y_{l-1} + \beta_2^{N_1 + \ldots + N_{l-1}} \beta_1^{1 + M_1 + \ldots + M_{l-1}} \sum_{i=0}^{N-1} \beta_2^i < 1\}$$

$$M_l = \min\{M \geq 1 | y_l + \beta_2^{N_1 + \ldots + N_l} \beta_1^{1 + M_1 + \ldots + M_{l-1} + M} < 1\}$$

By this construction we achieve:

Lemma 5.1 *With the notion from above we have for all $\beta_1, \beta_2 \in (0,1)$ with $\beta_1 + \beta_2 > 1$ and $k \geq 2$*

$$N_k \leq \max\{1, \log_{\beta_2}(\frac{\beta_1 + \beta_2 - 1}{\beta_1})\} =: B$$

and

$$V_t(\beta_1, \beta_2) - V_s(\beta_1, \beta_2) \leq (\frac{1}{\beta_1} + \frac{\beta_2}{1 - \beta_2})(\sqrt[1+B]{\beta_1})^{n_k}$$

$$\frac{\partial(V_t(x, \beta_2) - V_s(x, \beta_2))}{\partial x} \leq \frac{\beta_2}{1 - \beta_2} \mathcal{M}_k x^{\mathcal{M}_k - 1} - 1$$

Proof. By the definition of M_{k-1} we have

$$y_{k-1} + \beta_2^{\mathcal{N}_{k-1}} \beta_1^{\mathcal{M}_{k-1}-1} \geq 1$$

and by the definition of N_k we have

$$y_{k-1} + \beta_2^{\mathcal{N}_{k-1}} \beta_1^{\mathcal{M}_{k-1}} \frac{1-\beta_2^{N_k}}{1-\beta_2} < 1.$$

Hence

$$\beta_2^{\mathcal{N}_{k-1}} \beta_1^{\mathcal{M}_{k-1}} \frac{1-\beta_2^{N_k}}{1-\beta_2} < 1 - y_{k-1} \leq \beta_2^{\mathcal{N}_{k-1}} \beta_1^{\mathcal{M}_{k-1}-1}$$

and thus

$$\beta_2^{N_k} \geq \frac{\beta_1+\beta_2-1}{\beta_1}$$

given the bound on N_k state in our lemma. By this we immediately get

$$\mathcal{N}_k \leq Bk \leq B\mathcal{M}_k$$

and

$$n_k = \mathcal{N}_k + \mathcal{M}_k \leq (1+B)\mathcal{M}_k.$$

With $V_t(\beta_1,\beta_2) < 1 + \beta_1^{\mathcal{M}_k}(\beta_2/(1-\beta_2))$ and $V_s(\beta_1,\beta_2) \geq 1 - \beta_2^{\mathcal{N}_k}\beta_1^{\mathcal{M}_k-1}$ we now estimate

$$V_t(\beta_1,\beta_2) - V_s(\beta_1,\beta_2) \leq \beta_2^{\mathcal{N}_k}\beta_1^{\mathcal{M}_k-1} + \beta_1^{\mathcal{M}_k}\frac{\beta_2}{1-\beta_2} \leq \beta_1^{\mathcal{M}_k}(\frac{1}{\beta_1} + \frac{\beta_2}{1-\beta_2})$$

$$\leq (\frac{1}{\beta_1} + \frac{\beta_2}{1-\beta_2})(\sqrt[1+B]{\beta_1})^{(1+B)\mathcal{M}_k} \leq (\frac{1}{\beta_1} + \frac{\beta_2}{1-\beta_2})(\sqrt[1+B]{\beta_1})^{n_k}.$$

Calculating derivatives gives

$$V_s(x,\beta_2) = \sum_{j=0}^{k-1}(\beta_2^{\mathcal{N}_j} x^{\mathcal{M}_j} \sum_{i=0}^{N_{j+1}-1} \beta_2^i)$$

$$V_t(x,\beta_2) = 1 + x^{\mathcal{M}} \sum_{i=1}^{\mathcal{N}-1} \beta_2^i.$$

Hence

$$\frac{\partial V_s(x,\beta_2)}{\partial x} \geq 1 \text{ and } \frac{\partial V_t(x,\beta_2)}{\partial x} < \frac{\beta_2}{1-\beta_2}\mathcal{M}_k x^{\mathcal{M}_k-1}$$

given the estimate on the derivative. □

Now we are prepared to prove a proposition on the solution of algebraic equations $V_s = V_t$.

Proposition 5.1 *For all $\beta_1, \beta_2 \in (0,1)$ with $\beta_1+\beta_2 > 1$ there exits $n_k \longmapsto \infty$ and $s,t \in \Sigma_{n_k}$ such that the equation*

$$V_s(x,\beta_2) = V_t(x,\beta_2)$$

has a solution

$$x \in (\beta_1, \beta_1 + 2(\frac{1}{\beta_1} + \frac{\beta_2}{1-\beta_2})(\sqrt[1+B]{\beta_1})^{n_k}).$$

Proof. Let

$$f(x) = V_s(x, \beta_2) - V_t(x, \beta_2)$$

and

$$C_k = (\frac{1}{\beta_1} + \frac{\beta_2}{1-\beta_2})(\sqrt[1+B]{\beta_1})^{n_k}.$$

By our construction of the sequences $s, t \in \Sigma_{n_k}$ and lemma 5.1 $f(\beta_1) \leq C_k$. Moreover by lemma 5.1 $f'(x) < -1/2$ for all $x \in [\beta_1, \beta_1 + 2C_k]$ if k is large enough. By elementary calculus we have $f(x) = 0$ for $x \in (\beta_1, \beta_1 + 2C_k)$ proving our proposition. □

With the notions of this section proposition 3.1 is just a corollary to the last proposition.

Proof of proposition 3.1. Let $\beta_2 \in (1/4, 1/2)$ and $\beta_1 = 1/(4\beta_2)$. Obviously $\beta_1 + \beta_2 > 1$. Note that $T_s = T_t$ if $V_s = V_t$. Hence proposition 5.1 directly implies Proposition 3.1 with

$$c = 2(\frac{1}{\beta_1} + \frac{\beta_2}{1-\beta_2}) \text{ and } \lambda = \sqrt[1+B]{\beta_1}.$$

□

References

[1] K. Falconer, *Fractal Geometry - Mathematical Foundations and Applications*, Wiley, (New York, 1990).

[2] P. Erdös, *On a family of symmetric Bernoulli convolutions*, Amer.J.Math. 61 (1939), 974-975.

[3] P. Erdös, *On the smothness property of Bernoulli convolutions*, Amer.J.Math. 62 (1940), 180-186.

[4] J.E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. 30 (1981), 271-280.

[5] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems* , Cambridge University Press, (Cambridge 1995).

[6] S. Lally, *Random series in powers of algebraic integers: Hausdorff dimension of the limit distribution*, J.Lond.Math.Soc.II, 57, No. 3 (1998), 629-654.

[7] S.M.Ngau and Y. Wang, *Self-similar measures associated with IFS with non-uniform contraction ratios*, Preprint.

[8] J. Neunhäuserer, *Properties of some overlapping self-similar and some self-affine measures*, Acta Mathematica Hungarica vol. 93 1-2 (2001), 143-161.

[9] J. Neunhäuserer, *Number theoretical peculiarities in the dimension theory of dynamical systems*, Israel Journal of Mathematics 128 (2002), 267-283.

[10] M. Nicol, N. Sidorov and D. Broomhead, *On the finite structure of stationary measures*, J. Theor. Probab 15, No.3 (2002), 715-730.

[11] Y. Peres and B. Solomyak, *Absolutely continuous Bernoulli convolutions - a simple proof*, Math. Research Letters 3, No. 2 (1969), 231-239.

[12] Y. Peres, W. Schlag and B. Solomyak *Sixty years of Bernoulli convolutions*, Progress in probability 46 (2000), 39-65.

[13] Ya. Pesin, *Dimension Theory in Dynamical Systems - Contemporary Views and Applications*, University of Chicago Press, (Chicago 1997).

[14] , K. Simon and B. Solomyak, *On the dimension of self-similar sets*, Fractals 10 (2002), 59-65.

[15] B. Solomyak, *On the random series* $\sum \pm\lambda^i$ *(an Erdös problem)*, Ann. Math. 142 (1995), 611-625.