Dimensional theoretical properties of affine dynamical systems

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1 Introduction

Let (X,d) be a metric space and $T : X \longrightarrow X$ be a Borel measurable map. In the theory of dynamical systems we are interested in invariant sets like attractors and repellers and in invariant or especially ergodic measures with respect to the map T.[1] Let us give formal definitions of these objects.

Definition 1.1 A set $\Lambda \subseteq X$ is called T-**invariant** if $T(\Lambda) = \Lambda$. If there is is a neighbourhood U of Λ such that $\lim_{n\longrightarrow\infty} d(x,\Lambda) = 0\ \forall x \in U$, then Λ is called **attractive**. If in addition there exists $x \in \Lambda$ such that $\overline{\{x, Tx, T^2x, \dots\}} = \Lambda$ then Λ is called an **attractor** for T. If there is is an open set $V \subseteq X$ such that $\Lambda = \{x | T^n(x) \in V\ \forall n \geq 0\}$ then Λ is called a **repeller** for T. A Borel probability measure μ on X is called T-invariant if $\mu = \mu \circ T^{-1}$ and T-**ergodic** if in addition $T(B) = B \Rightarrow \mu(B) \in \{0,1\}$ holds for all Borel sets B in X.

Remark 1.1 *By Birkhoff's ergodic theorem ergodic measures describe the statistical long term behaviour of a dynamical system (see section 4.1. of [8]).*

It is a well known that invariant sets and ergodic measures often have a irregular and none smooth ("fractal") geometry. There are various kinds of "fractal" dimensions that allow us in some sense to measure the "size" of these objects and to discriminate them quantitatively.[2] We will introduce Hausdorff and box-counting dimension here.

Definition 1.2 Let (X,d) be a separable metric space X and Z be a subset of X. For $s \geq 0$ we define the s-**dimensional Hausdorff measure** $H^s(Z)$ of Z by

$$H^s(Z) = \lim_{\lambda\longrightarrow 0} \inf\{\sum_{i\in I}(\mathrm{diam} U_i)^s | I \text{ countable, } Z \subseteq \bigcup_{i\in I} U_i \text{ and } \mathrm{diam}(U_i) \leq \lambda\}.$$

The **Hausdorff dimension** $\dim_H Z$ of Z is given by

$$\dim_H Z = \sup\{s | H^s(Z) = \infty\} = \inf\{s | H^s(Z) = 0\}.$$

Let $N_\epsilon(Z)$ be the minimal number of balls of radius ϵ that are needed to cover Z. We define the **upper box-counting dimension** $\overline{\dim}_B Z$ resp. **lower box-counting dimension** $\underline{\dim}_B Z$ of Z by

$$\overline{\dim}_B Z = \overline{\lim}_{\epsilon\longrightarrow 0} \frac{\log N_\epsilon(Z)}{-\log \epsilon} \qquad \underline{\dim}_B Z = \underline{\lim}_{\epsilon\longrightarrow 0} \frac{\log N_\epsilon(Z)}{-\log \epsilon}.$$

[1] We recommend the book of Katok and Hasseblatt [8] for an introduction to the theory of dynamical systems.

[2] A good introduction to fractal geometry and dimension theory is the book of Falconer [5].

If $\underline{\dim}_B = \overline{dim_B}$ we call the value box-counting dimension, $\dim_B Z$, of Z. Now let μ be a Borel probability measure on X. We define the Hausdorff dimension of μ by

$$\dim_H \mu = \inf\{\dim_H Z | \mu(Z) = 1\}.$$

Remark 1.2 *All these dimensional theoretical quantities are invariant under bi-Lipschitz transformations. Thus dynamical systems that are bi-Lipschitz equivalent have the same dimensional theoretical properties.*

In developing a dimension theory of dynamical systems it is natural to ask the following questions:

Questions

(1) How can we calculate dimensional theoretical quantities in the context of dynamical systems?
(2) How are these quantities related to characteristics of the dynamics like Lyapunov exponents, entropy and pressure?
(3) Under what conditions resp. in which classes of systems do Hausdorff and box-counting of invariant sets coincide?
(4) Under what conditions resp. in which classes of systems there is an ergodic measure of full Hausdorff dimension on an given invariant set?

It seems to be well accepted by experts that these questions form a kind of guideline in the dimension theory of dynamical systems. For conformal dynamical systems we have a general theory answering all of these question. Especially we have theorems about conformal repellers and about hyperbolic sets for conformal diffeomorphisms (see Theorem 20.1., Theorem 22.1. and Theorem 22.2. of [16]). On the other hand the existence of different rates of contraction resp. expansion in different directions forces mathematical problems that are not solved in general these days. Affine dynamical systems given by pice-wise affine maps form $X \subseteq \mathbb{R}^n$ into $\mathbb{R}^n$ form the simplest class of non-conformal systems. They are easy to define but studying there dimensional theoretical properties provides substantial difficulties and it seems to be a long way to develop theoretical approaches that are strong enough to answer all of our questions for affine dynamical systems. This work contains a kind of overview about the field. We will write down no proofs but refer to the papers where the proofs can be found.
In section two we will present those theorems that we would consider as the main results in the area of dimension theory of affine dynamical systems. In section three we will present some new results we found in [14] and [15]. At the end in section four we will discuss a special topic. We will show in examples that the dimensional theoretical properties in a class of affine dynamical systems can considerably change because of number theoretical peculiarities of some parameter values (see [13],[15]).

2 Main results

Let $T_1, \ldots, T_p : \mathbf{D} \longrightarrow \mathbf{D}$ be affine contractions of a closed subset $\mathbf{D}$ of $\mathbb{R}^m$. Assume that the sets $T_i(\mathbf{D})$ are disjoint. From Hutchinson [7] we know there is an unique compact self-affine subset Λ of $\mathbf{D}$ satisfying:

$$\Lambda = \bigcup_{i=1}^{p} T_i(\Lambda).$$

Define a map T on $\bigcup_{i=1}^{p} T_i(\mathbf{D})$ by

$$T(x) = T_i^{-1}(x) \quad \text{if} \quad x \in T_i(\mathbf{D}).$$

Clearly T is a smooth expanding map. Λ is invariant and a repeller for T. Thus we call Λ a **self-affine repeller**.

There is one generic result about the dimension of large classes of self-affine sets.

Theorem 2.1 *Let $L_1, \ldots, L_p$ be linear contractions of $\mathbb{R}^m$ with $||L_i|| < 1/2$ for $i = 1, \ldots, p$ and let $b_1, \ldots, b_p \in \mathbb{R}^m$. If Λ is the compact self-affine set satisfying*

$$\Lambda = \bigcup_{i=1}^{p} (L_i(\Lambda) + b_i)$$

then $\dim_B \Lambda = \dim_H \Lambda = d$ holds for almost all $(b_1, \ldots, b_p) \in (\mathbb{R}^m)^p$ in the sense of mp-dimensional Lebesgue measure and d is independent of $(b_1, \ldots, b_p)$.

Remark 2.1 *Falconer [6] proved this theorem in the case $||L_i|| < 1/3$ and Solomyak [22] extended the proof to the case $||L_i|| < 1/2$. Moreover Solomyak showed that the statement does not longer hold if we replace $1/2$ by $1/2 + \delta$ with $\delta > 0$.*

Remark 2.2 *There is a way approximate the constant d using the singular value function of the compositions of the maps $L_1, \ldots L_p$ (see [6] or [5]).*

Remark 2.3 *Theorem 2.1 leaves many questions open. First of all the question about the existence of an ergodic measure of full Hausdorff dimension remains completely open. Moreover one would like to have results about natural subclasses of self-affine repeller that fall into the exceptional set of 2.1 and about classes of self-affine repellers with larger expansion rates.*

Let us discuss a very natural family of self-affine repellers that is completely understood today and proved to fall in the exceptional class of 2.1 .
Given integers $l \geq m \geq 2$ choose a set A of pairs of integers (i, j) with $0 \leq i < l$ and $0 \leq j < m$. Denote the cardinality of A by a. Now let T_k for $k = 1 \ldots a$ be affine maps given in the following way: if k enumerates the element $(i, j) \in A$ then let

$$T_k([0, 1]^2) = [i/l, (i + 1)/l] \times [j/m, (j + 1)/m].$$

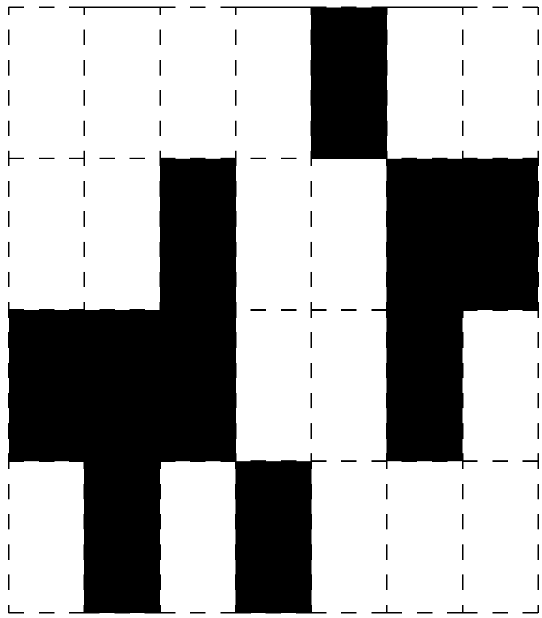

Figure 2: The images of the affine maps inducing a self-affine carpet with $l = 8$ $m = 4$ and $A = \{(4,0),(2,1),(6,1),(7,1),(0,2),(1,2),(2,2),(5,2),(1,3),(3,3)\}$

Let Λ_A be the self-affine set generated by these affine contractions. A set of this type is known as **general Sierpinski carpet**. We remark that Λ_A viewed as a subset of the Torus is invariant under the toral endomorphism given by

$$\hat{T} : (x, y) \longrightarrow (lx, my) \mod 1.$$

Dimensional theoretical questions are answered by the following theorem of McMullen [11] and Bedford [2].

Theorem 2.2 *Let t_j be the number of those i for which $(i, j) \in A$ and let r be the number of those j for which there is some i such that $(i, j) \in A$. We have*

$$\dim_H \Lambda_A = \log_m\Big(\sum_{j=0}^{m-1} t_j^{\log_l m}\Big) \quad \text{and} \quad \dim_B \Lambda_A = \log_m r + \log_l (a/r).$$

Moreover there exists a Bernoulli measure of full Hausdorff dimension on Λ_A.

Remark 2.4 *It is easy to see that a Bernoulli measure on the carpet is in fact an ergodic measure with respect to the map $\hat{T}$ on the torus and the expanding map T associated with the affine contractions.*

Remark 2.5 *Note that Theorem 2.2 implies that the Hausdorff and box-counting dimension of a general Sierpinski carpet coincide if and only if the carpet is self-similar (l=m) or the number of rectangles is constant or zero in every raw ($t_j = 0$ or $t_j = const.$ for all j).*

Kenyon and Peres [9] extended Theorem 2.2 to analogous subsets of higher dimensional cubes, which they called **self-affine Sierpinski sponges**. Using this result and an approximation argument they where also able to show the following theorem:

Theorem 2.3 *Let T be a linear endomorphism of the d-Torus with integer eigenvalues. If K is a compact T-invariant set then there exists a T-ergodic measure of full Hausdorff dimension on K.*

Dimensional theoretical properties of **attractors of endomorphisms with singularities** were studied by Schmeling and Troubetzkoy ([23], [20]). The main example of Schmeling and Troubetzkoy is the class of **Belykh maps** given by piecewise affine transformations

$$f^k_{\beta,\tau}(x,y) = \{ \begin{array}{ll} (\beta x + (1-\beta), \tau y + (1-\tau) & \text{if} \quad y \geq kx \\ (\beta x - (1-\beta), \tau y - (1-\tau) & \text{if} \quad y < kx \end{array}$$

with $(\beta,\tau,k) \in H := \{(\beta,\tau,k)|\beta \in (0,1),\ k \in (-1,1),\ \tau \in (1, 2/(|k|+1)\}$ on the square $[-1,1]^2$. Let $U = \{(x,y)|x \leq ky\}$, $V = \{(x,y)|y \leq kx\}$ and

$$Q = \{(x,y)|f^n(x,y) \notin \partial U \cup \partial V \ \forall n \geq 0\}.$$

The set

$$\Lambda^k_{\beta,\tau} := \overline{\bigcap_{n\geq 0} f^n(Q)}$$

is called **Belykh attractor** (see [3] and [20]).

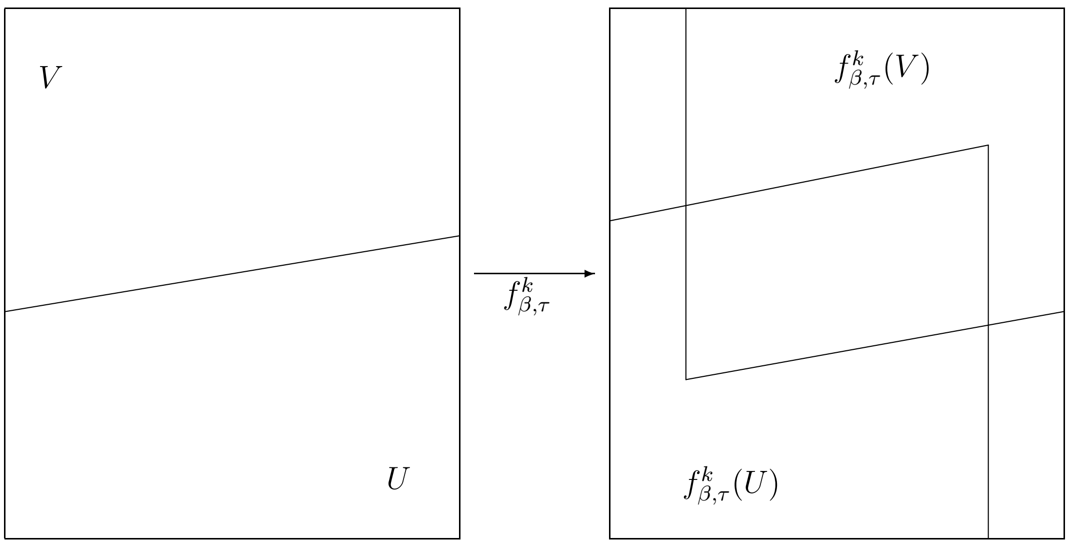

Figure 2: The Belykh map

Theorem 2.4 *For almost all $(k,\beta,\tau) \in H$ with $\beta < 0.649$ we have*

$$\dim_H \mu_{SRB} = \dim_H \Lambda^k_{\beta,\tau} = \dim_B \Lambda^k_{\beta,\tau} = \min\{1 - \frac{\log\tau}{\log\beta}, 2\}$$

where μ_{SRB} is the Sinai-Ruelle-Bowen measure for the system $(\Lambda^k_{\beta,\tau}, f^k_{\beta,\tau})$.

Remark 2.6 *This theorem is the well known Kaplan-Yorke conjecture (Lyapunov dimension=information dimension) for the special case of the Belykh attractors (see remark (ii) in [20]).*

Remark 2.7 *In [23] a formula for the dimension of the Sinai-Ruelle-Bowen measure is proved for a broader class of hyperbolic endomorphisms with singularities.*

3 Some new results

Now we present some new results we found in [13] and [15]. Let $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2) \in (0,1)^4$ and assume $\beta_1 + \beta_2 \geq 1$ and $\tau_1 + \tau_2 < 1$. We define two affine contractions $T_{1,\vartheta}$ and $T_{2,\vartheta}$ of the square $[-1,1]^2$ by

$$T_{1,\vartheta}(x,z) = (\beta_1 x + (1-\beta_1), \tau_1 z + (1-\tau_1))$$

$$T_{2,\vartheta}(x,z) = (\beta_2 x - (1-\beta_2), \tau_2 z - (1-\tau_2)).$$

Let Λ_ϑ be the corresponding self-affine set and T_ϑ be the transformation associated with these affine maps (see section 2).

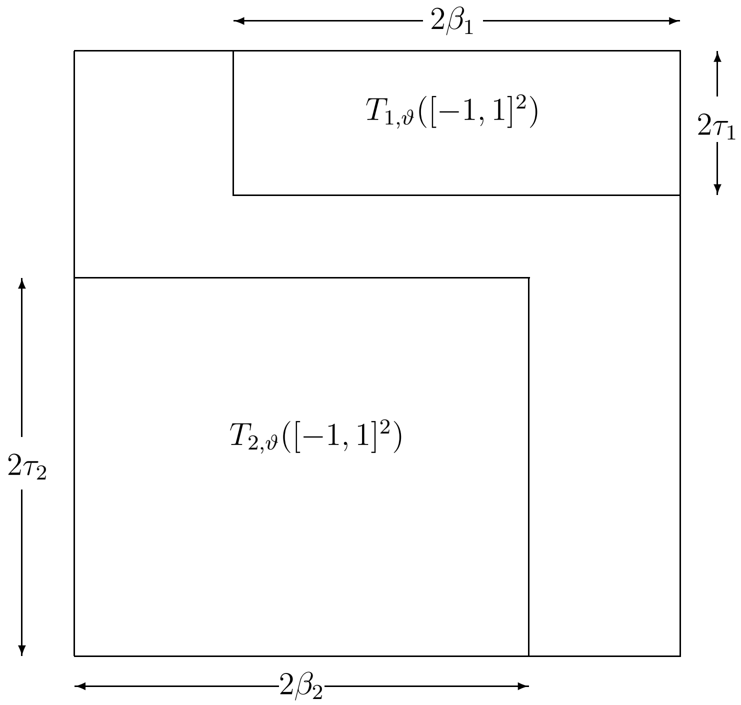

Figure 3: The transformations $T_{1,\vartheta}$ and $T_{2,\vartheta}$ on $[-1,1]^2$

Theorem 3.1 *For almost all* $\vartheta \in \{(\beta_1, \beta_2, \tau_1, \tau_2) \in (0, 0.649)^2 \times (0,1)^2 \mid \beta_1 + \beta_2 \geq 1, \quad \tau_1 + \tau_2 < 1\}$ *we have*

$$\dim_H \mu_\vartheta = \dim_H \Lambda_\vartheta = \dim_B \Lambda_\vartheta = d + 1,$$

where μ_ϑ *is an ergodic measure with respect to* $(\Lambda_\vartheta, T_\vartheta)$ *and* d *is the solution of* $\beta_1 \tau_1^x + \beta_2 \tau_2^x = 1$.

Remark 3.1 *The ergodic measure* μ_ϑ *is constructed as a Bernoulli measure with weights* $(\beta_1\tau_1^d, \beta_2\tau_2^d)$ *on the self-affine set* Λ_ϑ *(see [14]).*

Remark 3.2 *The proof of Theorem 3.1 given in [14] has two main parts. We study certain self-similar measures on the real line which are the projections of self-affine measures on the sets* Λ_ϑ *and we apply the general dimension theory for ergodic measures developed in [10] and [4] in the context of systems with singularities.*

Remark 3.3 *Consider the symmetric self-similar sets* $\Lambda_{\beta,\tau} := \Lambda_{(\beta,\beta,\tau,\tau)}$ *and let* $T_{\beta,\tau} = T_{(\beta,\beta,\tau,\tau)}$. *It follows from [21] together with Theorem II of of [14] that for almost all* $\beta \in (0,5,1)$ *and all* $\tau \in (0,0.5)$ *the identity*

$$\dim_H \mu_{\beta,\tau} = \dim_H \Lambda_{\beta,\tau} = \dim_B \Lambda_{\beta,\tau} = \frac{\log(2\beta)}{\log(\tau)} + 1$$

holds. Here $\mu_{\beta,\tau}$ *is the equal-weighted Bernoulli measure on* $\Lambda_{\beta,\tau}$ *which is* $T_{\beta,\tau}$*-ergodic. This identity has been shown before by Pollicott and Weiss [18] under the assumption that* β *is a Garsia-Erdös number.*[3]

Now we consider a very simple but nevertheless interesting family of attractors of piecewise affine transformations. Let $\beta_1, \beta_2 \in (0,1)$ and define $f_{\beta_1,\beta_2} : I\!R \times [-1,1] \longmapsto I\!R \times [-1,1]$ by

$$f_{\beta_1,\beta_2}(x,y) = \left\{ \begin{array}{ll} (\beta_1 x + (1-\beta_1), 2y-1) & \text{if} \quad y \geq 0 \\ (\beta_2 x - (1-\beta_2), 2y+1) & \text{if} \quad y < \ 0 \end{array} \right.$$

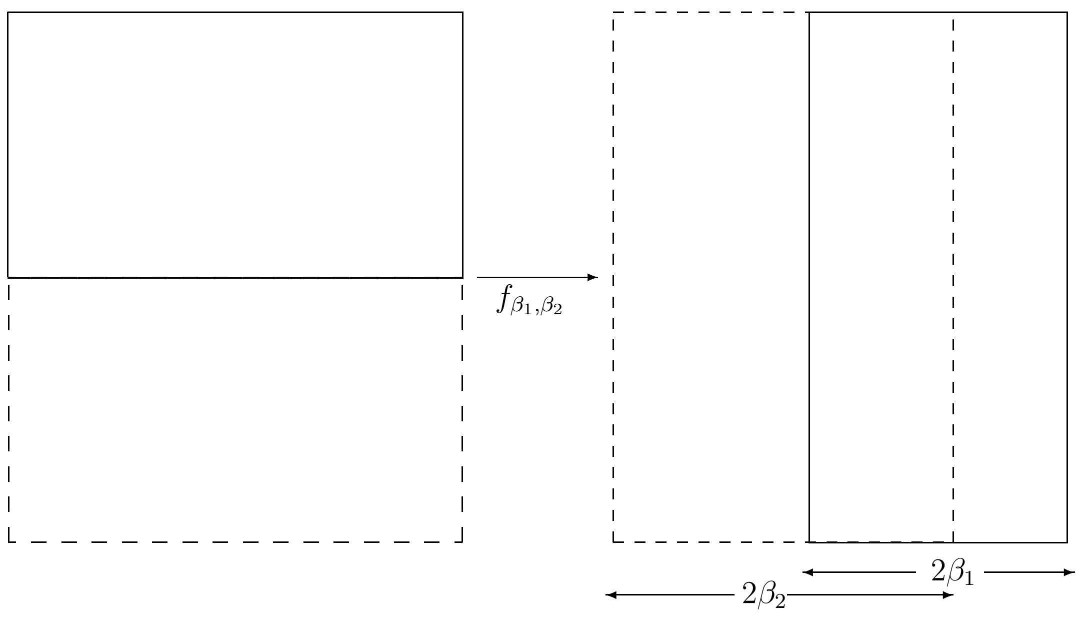

Figure 4: The action of f_{β_1,β_2} on the square $[-1,1]^2$ where $\beta_1 + \beta_2 > 1$

We call this family of maps **generalised Baker's transformations** because if we set $\beta_1 = \beta_2 = 0.5$ we get the well known classical Baker's transformation.

[3]This means $\exists\ C > 0\ \forall x \in I\!R$: $\text{card}\{(s_0,\ldots,s_{n-1}) \in \{-1,1\}^n | \sum_{k=0}^{n-1} s_k\beta^k \in [x, x+\beta^n)\} \leq C(2\beta)^n\ \forall n \geq 1$.

If $\beta_1 + \beta_2 < 1$ the attractor of f_{β_1,β_2} is a product of a cantor set with the interval $[-1, 1]$ and the dimensional theoretical properties of the system are easy to understand (see section 23 of [16]). If $\beta_1 + \beta_2 \geq 1$ the attractor is obviously the whole square $[-1, 1]^2$ which has Hausdorff and box-counting dimension two. The interesting problem in this situation is if there exists an ergodic measure of full Hausdorff dimension. In [15] we proved the following result:

Theorem 3.2 *For almost all $(\beta_1, \beta_2) \in (0, 0.649)$ with $\beta_1 + \beta_2 \geq 1$ and $\beta_1\beta_2 \geq 0.25$ there is an ergodic measure of full dimension for $([-1, 1], f_{\beta_1,\beta_2})$. But if are given $(\beta_1, \beta_2) \in (0, 1)$ with $\beta_1 + \beta_2 \geq 1$ and $\beta_1\beta_2 < 0.25$ then we have $\sup\{\dim_H \mu | \mu \quad f_{\beta_1,\beta_2}\text{-ergodic}\} < 2$.*

Remark 3.4 *This example shows that it is not always possible to find the Hausdorff dimension of an invariant set by constructing an ergodic measure of full Hausdorff dimension. But nevertheless this is at least generically possible in many situations (compare with Theorem 2.2, 2.4 and 3.1).*

Remark 3.5 *Roughly speaking the reason why there is not always an ergodic measure of full Hausdorff dimension here is that one can not maximise the stable and the unstable dimension (the dimension of conditional measures on partitions in stable resp. unstable directions) at the same time. In another context this praenomen was observed before by Manning and McClusky [12].*

Remark 3.6 *Consider the* **Fat Baker's transformation** $f_\beta := f_{\beta,\beta}$ *where $\beta \in (0.5, 1)$. It follows from the work of Alexander and Yorke [1] together with [21] that for almost all $\beta \in (0.5, 1)$ we have $\dim_H \mu_{SRB} = 2$ where μ_{SRB} is the Sinai-Ruelle-Bowen measure for the system $([-1, 1]^2, f_\beta)$. This means that in the symmetric situation we generically have an ergodic measure of full dimension.*

4 Number theoretical peculiarities

In this section we want to show in examples that number theoretical peculiarities of parameter values can considerably change the dimensional theoretical properties in a class affine dynamical systems. This point is of some interest because it gets clear how difficult it is to develop a general dimension theory of dynamical systems and that one would expect most results to hold only generically.

All our examples here are related to special algebraic integers which we now define.

Definition 4.1 A **Pisot-Vijayarghavan number** (short: PV number) is the root of an algebraic equation whose conjugates lie all inside the unit circle.

Remark 4.1 *Salem [19] showed that the set of PV numbers is a closed subset of the reals and that 1 is an isolated element.*

Remark 4.2 *In our context we are interested in numbers $\beta \in (0.5, 1)$ such that β^{-1} is a PV number. We list some examples including all reciprocals of PV numbers with minimal polynomial of degree two and three and a sequence of such numbers decreasing to* 0.5 *(see [1]).*

$x^2 + x - 1$	$(\sqrt{5} - 1)/2$
$x^3 + x^2 + x - 1$	$0.5436898\ldots$
$x^3 + x^2 - 1$	$0.754877\ \ldots$
$x^3 + x - 1$	$0.6823278\ldots$
$x^3 - x^2 + 2x - 1$	$0.5698403\ldots$
$x^4 - x^3 - 1$	$0.7244918\ldots$
$x^n + x^{n-1} \ldots + x - 1$	$r_n \longrightarrow 0.5$

Table 1: *Reciprocals of PV numbers*

Now we consider the Fat Baker's transformations $([-1, 1]^2, f_\beta)$ (defined in remark 3.6) and the symmetric self-affine repeller $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$ (defined in 3.3).

Theorem 4.1 *Let $\beta \in (0.5, 1)$ be the reciprocal of a PV number and $\tau \in (0, 0.5)$. We have:*
(1) $\{\dim_H \mu | \mu\ f_\beta\text{-ergodic}\} < 2$.
(2) There is no Bernoulli measure of full dimension for the system $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$.
(3) $\dim_H \Lambda_{\beta,\tau} < \dim_B \Lambda_{\beta,\tau}$

Remark 4.3 *We proved statement (1) and (2) in [13]. Statement (1) is an extension of the result of Alexander and York [1] that states that the Sinai-Ruelle-Bowen measures does not have Rényi dimension two in our context. Pollicott and Wise claimed that statement (3) follows for small τ from the work of Przytycki and Urbanski [17]. Because we were not able to see that this is true we wrote down an independent proof of (3) in [15].*

Remark 4.4 *Statement (1) and (3) show that the existence of an ergodic full dimension Hausdorff dimension and the identity of box-counting and Hausdorff dimension can drop in the context of affine dynamical because of number theoretical peculiarities.*

Remark 4.5 *We do not know if there exists an ergodic measure of full Hausdorff dimension for the systems $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$ and can not calculate $\dim_H \Lambda_{\beta,\tau}$ in the case that $\beta \in (0.5, 1)$ is the reciprocal of a PV number. Statement (2) only shows that the we can not calculate $\dim_H \Lambda_{\beta,\tau}$ by means of Bernoulli measures in this situation.*

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